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HOW TO STIMULATE RICH INTER-ACTIONS AND REFLECTIONS IN ONLINE MATHEMATICS TEACHER EDUCATION?

Mario Sánchez Aguilar
PhD Dissertation
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Taking into consideration the information obtained through the literature reviews, but also drawing on my practical experiences in the design and implementation of online in-service courses for mathematics teachers, two research questions are formulated: (1) what are the characteristics of the online interactions that promote emergence of mathematics teachers' reflections? and (2) which non-human elements of an online course promote the emergence of mathematics teachers' reflections? These two questions are investigated through the design, implementation and analysis of the outcomes of two online in-service courses for mathematics teachers.

The results indicate that the evaluative acts and the challenging acts are crucial for the emergence of mathematics teachers' reflections. They also indicate that theoretical concepts from mathematics education research are resources that help to trigger the emergence of mathematics teachers' reflections.

The dissertation concludes with a discussion of the theoretical implications and practical applications of the research results. The main contributions of this research are: (1) a characterisation of the concept of reflection which allows to transform such a cognitive process into a researchable and identifiable entity within an online setting; (2) the identification of the communicative characteristics of an online interaction that favour the emergence of mathematics teachers' reflections; and (3) the identification of elements in the design of an online course that promote the emergence of mathematics teachers' reflections.

Mario Sánchez Aguilar, Denmark, 2010

R O S K I L D E   U N I V E R S I T Y

IMFUFA, DEPARTMENT OF SCIENCE, SYSTEMS AND MODELS

HOW TO STIMULATE RICH INTERACTIONS AND REFLECTIONS IN ONLINE MATHEMATICS TEACHER EDUCATION?

PhD Dissertation in Mathematics Education Research

Mario Sánchez Aguilar

Re: Equipo 1. "Tecnica del comando factor" de Mónica Lorena Micelli - Tuesday, 25 de November de 2008, 16:20
Re: Equipo 1. "Tecnica del comando factor" de Mónica Lorena Micelli - Tuesday, 25 de November de 2008, 16:33
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Re: Equipo 1. "Tecnica del comando factor" de Mónica Lorena Micelli - Saturday, 29 de November de 2008, 03:21
Re: Equipo 1. "Tecnica del comando factor" de Rebeca Flores García - Saturday, 29 de November de 2008, 16:56

Supervisor: Dr. Morten Blomhøj, Roskilde University, Denmark

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How to stimulate rich interactions and reflections in online mathematics teacher education?
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Illustration on the cover: Snapshot of an asynchronous discussion forum

For Idania

How to stimulate rich interactions and reflections in online mathematics teacher education?

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Denmark, 2010
Mario Sánchez Aguilar

Abstract

This dissertation reports on a theoretical and empirical study of the emergence of mathematics teachers' reflections in online in-service teacher education.

The study begins by presenting two comprehensive reviews of the research literature on mathematics teacher education. In the first review the current research trends in this field are identified, among which reflective thinking and online teacher education are included. Besides providing an overview of the current state of research on mathematics teacher education, the first review helps to justify the scientific relevance of this research. The second review focuses on clarifying how the concept of reflection is defined in the research literature and why it is considered as particularly relevant to the professional development of mathematics teachers.

Taking into consideration the information obtained through the literature reviews, but also drawing on my practical experiences in the design and implementation of online in-service courses for mathematics teachers, two research questions are formulated: (1) what are the characteristics of the online interactions that promote emergence of mathematics teachers' reflections? and (2) which non-human elements of an online course promote the emergence of mathematics teachers' reflections?

These two questions are investigated through the design, implementation and analysis of the outcomes of two online in-service courses for mathematics teachers. The courses contain some special elements aimed at fostering interactions and reflections among the teachers. These elements are asynchronous discussion forums; "notes of reflections” which are written case studies in which a fictional situation is described; and heterogeneous working groups were the members of the groups have different opinions on the issues being discussed. Such elements create a setting in which the study of online interactions and reflections is facilitated.

To answer the research question (1) a characterisation of the communicative acts that are present in online interactions where teachers' reflections appear is carried out. Then, the common characteristics that are considered as key to the emergence of teachers' reflections are located. The results indicate that the evaluative acts and the challenging acts are crucial for the emergence of mathematics teachers' reflections.

To answer the research question (2) a connection between the resources that are part of an online course and the reflections that emerge within the online course are established. Through such connection the resources that influence the formation of teachers' reflections are located. It is found that theoretical concepts from mathematics education research are resources that help to trigger the emergence of mathematics teachers' reflections.

The dissertation concludes with a discussion of the theoretical implications and practical applications of the research results. The main contributions of this research are: (1) a characterisation of the concept of reflection which allows to transform such a cognitive process into a researchable and identifiable entity within an online setting; (2) the identification of the communicative characteristics of an online interaction that favour the emergence of mathematics teachers' reflections; and (3) the identification of elements in the design of an online course that promote the emergence of mathematics teachers' reflections.

Resumé

Denne afhandling beretter om en teoretisk og empirisk undersøgelse af fremkomsten af refleksioner hos matematiklærere som deltager i online efteruddannelse.

Undersøgelsen starter med to omfattende reviews af forskningslitteraturen inden for matematiklæreruddannelse. I det første review identificeres aktuelle forskningstendenser indenfor dette område, herunder refleksiv tænkning og online læreruddannelse. Udover at give et overblik over forskningens aktuelle tilstand indenfor matematiklæreruddannelse, bidrager det første review til at begrunde relevansen af afhandlingens forskning. Det andet review fokuserer på at klargøre hvordan begrebet refleksion defineres i forskningslitteraturen og hvorfor det opfattes som særlig relevant for matematiklæreres professionelle udvikling.

På basis af informationerne indsamlet i disse reviews såvel som mine egne praktiske erfaringer med at designe og implementere online efteruddannelseskurser for matematiklærere, formuleres to forskningsspørgsmål: (1) hvilke karakteristika har online interaktioner, der fremmer fremkomsten af matematiklæreres refleksioner? Og (2) hvilke ikke-menneskelige elementer i et online kursus fremmer fremkomsten af matematiklæreres refleksioner?

Disse to spørgsmål undersøges gennem design, implementering og analyse af resultaterne af to online efteruddannelseskurser for matematiklærere. Kurserne indeholder særlige elementer rettet mod at fostre interaktion mellem og refleksioner hos lærerne. Disse elementer er: asynkrone diskussionsfora; "refleksionnoter," som er skriftlige casestudier, der beskriver fiktive situationer; og, heterogene arbejdsgrupper, hvor

gruppemedlemmerne har forskellige meninger om de diskuterede emner. Sådanne elementer skaber et miljø, der faciliterer en undersøgelse af online interaktion og refleksioner.

For at svare på forskningsspørgsmål (1) karakteriseres de kommunikative handlinger, der er tilstede ved de online interaktioner, hvor lærernes refleksioner optræder. Derefter lokaliseres de fælles karaktertræk, der opfattes som nøglen til fremkomsten af lærernes refleksioner. Resultaterne indikerer at evaluerings- og udfordrende handlinger er afgørende for fremkomsten af matematiklæreres refleksioner.

For at svare på forskningsspørgsmål (2), etableres en sammenhæng mellem ressourcerne som indgår i et online kursus og lærernes refleksioner som fremkommer i kurset. Herigennem lokaliseres de ressourcer som påvirker sammensætningen af lærernes refleksioner. Det viser sig at teoretiske begreber fra forskning i matematikkens didaktik er ressourcer, som hjælper til at sætte fremkomsten af matematiklæreres refleksioner i gang.

Afhandlingen slutter med en diskussion af de teoretiske implikationer og praktiske anvendelser af forskningsresultaterne. Forskningens hovedbidrag er: (1) en karakterisering af begrebet refleksion, som tillader at dennne kognitive proces omdannes til en forsknings- og identificerbar entitet i et online miljø; (2) identificeringen af de kommunikative karakteristika ved en online interaktion, som begunstiger fremkomsten af matematiklæreres refleksioner; og (3) identificeringen af de elementer i designet af et online kursus som fremmer fremkomsten af matematiklæreres refleksioner.

Contents

1. Introduction

I will begin this introduction by presenting my own background as mathematics educator, since it will provide the reader with a useful context to clarify the origin and motivation for the research presented in this PhD dissertation.

1.1 Personal background

When I was 18 years old I started working as a mathematics teacher in an adult education centre located in the western-pacific area of Mexico. I just loved the experience. Two years later I started a bachelor in mathematics, partly inspired by the desire to become a better mathematics teacher. During and after the bachelor studies I continued working as a mathematics teacher, not only in adult education but also in secondary education and even at the University level. I had eight years of experience as a teacher when I travelled to Mexico City and started my graduate studies in mathematics education. At that time I thought that through these studies I would learn "techniques and recipes" to become a better teacher.

I started my graduate studies in mathematics education but I did not find the "techniques and recipes" I expected. However, I found a fascinating new world. It was an introduction to the mathematics education research that made me see the teaching and learning of mathematics from a completely different perspective. I was particularly interested in the courses on the use of technology (mathematical software, graphic calculators, Internet), perhaps because I never had real access to it before[1]. I participated with a lot of enthusiasm in such courses. Asuman Oktaç[2] taught one of these courses. The aim of the course was to observe and discuss the development of an online course for in-service mathematics teachers. She was invited to design and apply the course in an online mathematics teacher education program called CICATA[3], and she decided to invite her graduate students to observe it. This was my first encounter with the online mathematics teacher education[4].

I found it very interesting to see how teachers expressed and communicated mathematical ideas in an environment in which the tools to communicate mathematical symbols and graphs were limited. In fact in my master's thesis I analysed how teachers studying in this online setting communicate mathematical ideas when solving mathematical problems (see Sánchez, 2003).

1.1.1 The motivation for this research

Just after completing my graduate studies I returned to my work as a mathematics teacher. A few months later I received the offer to join the mathematics teacher educators staff of the CICATA program where I got the empirical data for my master's thesis. At that moment I began my

[1] As a student I never could afford to buy a graphic calculator nor a computer. My first encounter with a computer was at the end of my bachelor studies. I owned a computer and surfed the Internet for the first time, just some months before starting my master studies.

[2] www.matedu.cinvestav.mx/aoktac.html

[3] www.matedu.cicata.ipn.mx

[4] When I use the term "online mathematics teacher education" I refer to a specific type of mathematics teacher education in which the content and activities of the courses are delivered via the Internet. In other words, the participants in this type of education do not meet physically to interact and discuss. All the interaction and communication is carried out by using the Internet and related communication tools such as email, discussion forums, audio and video conferencing.

career as an online mathematics teacher educator. The rest of the staff was composed of a group of young academics led by an experienced mathematics educator (see figure 1).

Figure 1. Part of the staff of the online mathematics teacher education program CICATA in Mexico City. Year 2005. From left to right: Elizabeth, Mario (myself), Gisela, Alejandro, Apolo and Gabriel.

All the academic staff of this educational program had carried out graduate studies (master's degree or PhD) in mathematics education research, but virtually none (except our leader) had experience as a teacher educator. My point here is that our approach to mathematics teacher education has been very innovative but primarily driven by intuitive ideas and pragmatic developments.

The CICATA program offers master's studies in mathematics education. The studies are aimed at in-service mathematics teachers from all over Latin America[5]. One of my main responsibilities within this educational program is to design and implement online courses that are part of the curricula of the master's studies. Through my experience I found that these courses were not always equally "successful"; i.e. teachers sometimes interacted very lively and reflected upon their teaching practice and how to improve it. Sometimes the same teachers seemed less engaged and more distant. In this context the question that drove this research arose:

> "How to stimulate rich interactions and reflections in online mathematics teacher education?"

The previous question hovered above my head a couple of years. Then I had the opportunity to travel to Denmark and start my PhD studies in mathematics education research at Roskilde University.

1.1.2 Studying in Denmark

In July 2004, the *10th International Congress on Mathematical Education* (ICME-10) was held in Copenhagen, Denmark. I did not attend the conference, but Ricardo Cantoral[6], one of my former mathematics education teachers did it. During the conference, Ricardo Cantoral met Mogens Niss[7] from Roskilde University (RUC) in Denmark. Mogen Niss

[5] The online-based education provides teachers with a flexible study schedule that allows them to study a master's degree without having to quit their jobs as mathematics teachers. The online-based education also eliminates geographic barriers, since is not necessary for the teachers to be physically present in the courses. They can participate in this educational program from anywhere in the world.

[6] http://cimate.uagro.mx/cantoral/

[7] http://forskning.ruc.dk/site/research/niss_mogens_allan(2077)/

was in charge of the organisation of the ICME conference in Copenhagen. Ricardo discussed with Mogens the possibility of sending Mexican students to study mathematics education research at RUC. Mogens agreed. When Ricardo Cantoral returned to Mexico, he told me that there was this possibility of studying in Denmark and that I should seize it. I should get in contact with Mogens Niss and also try to obtain funding to carry out the studies.

When I contacted Mogens Niss and showed him a sketch of my academic project he told me that I could develop it at RUC. On the other hand, I got economic financing through grants from the European Union, the Mexican Ministry of Education, and the National Polytechnic Institute of Mexico. Thus, my PhD studies at RUC are an example of the "side effects" that an international conference in mathematics education may have.

Studying at RUC transformed me as a mathematics educator. I was fortunate to develop my project under the supervision of Morten Blomhøj[8], who contributed to challenge and change many of the conceptions and ideas that I had about the discipline and the way research is done. In this introduction I tried to reflect some aspects of these learning and transformation processes that I experienced in Denmark.

One of the first things I learned was that the question "How to stimulate rich interactions and reflections in online mathematics teacher education?" was not really a research question. It was necessary to narrow it down, to make it more precise and thereby researchable. To reformulate this initial question into a research question has been one of the most challenging tasks during my PhD.

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[8] http://forskning.ruc.dk/site/research/blomhoej_morten(2085)/

1.2 The first research question

My research project is based on several assumptions. One of them is that teachers actually produce reflections when interacting in an online setting. I mean, through my practice as an online teacher educator, I have observed teachers while they consciously reflected on the way they teach mathematics, about their mathematical knowledge and on the behaviour of their students. My research was driven by the interest in identifying the factors that favour the emergence and development of such reflections.

My practical experience as a designer of online courses also indicated to me that there might be a relationship between the nature of the online interactions and the emergence and quality of teachers' reflections. That is, it seemed to me that the way in which teachers interacted during the online courses influenced teachers' reflections. Particularly it seemed as if teachers' reflections were more likely to appear during "lively" online interactions. That is, during interactions where teachers actively exchange and discuss ideas, questions and opinions related to a particular topic.

My first research question was based on the assumption that there is a kind of online interaction that promotes teachers' reflections. This is an assumption based on my own observations as online teacher educator. However, it was not clear to me what characterise such kind of interactions. Thus, the first research question that I formulated was:

> **(1) What are the characteristics of the online interactions that promote the emergence of mathematics teachers' reflections?**

This question *is* a research question. The original question "How to stimulate rich interactions and reflections in online mathematics teacher education?" was too broad to be considered as a researchable question. It did not have a clear starting point because, where and how should we look for the possible factors that stimulate a reflection? In addition, it was not possible to produce a precise answer to this question. The "how to?" question was too broad to be answered specifically.

Question 1 is considered a research question because it is more narrow and precise (although this is not the only characteristic that makes me regard it as a research question). The question can be answered specifically by identifying the characteristics of the interactions favouring the emergence of reflections.

Here I need to make an important clarification. It is likely that the way I am presenting my ideas could produce a feeling of simplicity in the reader associated with my research that did not exist. For example, the phrase "Thus, the first research question I formulated was" could be interpreted as if the process of establishing the first question was straightforward and without complications. But it was not at all like that! Although my research was initially motivated by problems experienced in my practice as an online teacher educator (a problem-driven research, as Arcavi (2000) calls it), the formulation of the first research question was a non-linear process in which theory and practice were intertwined. The practice served as a supplier of problems that are important to address (important from a practical and personal point of view). However, not until I started studying the specialised literature, I was able to understand what aspects of these practical problems could be *researchable* and *relevant* to the community of mathematics teacher educators. By *researchable* I am refering to the possibility of finding theoretical concepts which would allow me to conceptualise the key components of the problems associated with my practice as online teacher educator. Finding out whether a research question is *relevant* or not means to determine if the study of the questions that are important to me personally could also contribute to the development of the field of mathematics teacher education research.

Thus, based on the experience I got from developing this research, I have the impression that it is quite difficult for a newcomer to the field of mathematics education research to establish a decent research question, at least from the very beginning of his/her PhD project. Before it is necessary to engage in a two-way process in which the nature of your practical concerns can give you an idea of what type of articles and theoretical constructs you should seek and study. Then such articles and theoretical constructs help to shape your practical concerns and turn them into researchable questions. This two-way process involving interaction between theory and practice not only serves to shape the research questions. Such process also informs other aspects of the research method.

After this clarification, I will continue my discussion of the relevance of the first research question:

The research question 1 is relevant in a personal context because it is intended to validate (or refute) my experience-based hypothesis about the existence of a relationship between certain types of interactions and the emergence of teachers' reflections. However, the research question is also scientifically relevant as it addresses one of the key components in the development of mathematics teachers, namely the *reflective thinking*[9]. Particularly, the answer to research question 1 involves the empirical identification of the types of reflections that mathematics teachers are likely to experience. Such identification may foster discussions about the *types* of reflections that should be considered as relevant for the

[9] In chapter 2, where the main research trends in mathematics teacher education are discussed, reflective thinking is classified as one of those main research trends. Furthermore, in chapter 3 it is discussed why reflection is seen as a key component in the development of mathematics teachers.

development of mathematics teachers. Moreover, the answer to research question 1 will support the development of the emerging research area *online teacher education*[10], by providing methodological information about the manner in which teachers' reflections can be detected in an online setting.

Question 1 is a researchable and well-defined research question. However, in order to answer it, it is necessary to develop a process. The process to which I refer is the *research method*. This method is discussed in the next section.

1.3 Research method (Part 1)

My conception of what a research method is was significantly modified during my PhD studies at Roskilde University (RUC). Previously, I believed that the "method" was a sort of ingredient that should be included in any dissertation, and which *only* explained how the empirical data were selected and analysed (in the case of an empirical research). I had a sort of recipe conception of what a research method was.

At RUC I discovered that a research method is a comprehensive process that includes all the choices and actions that a researcher carries out in order to establish a research question, and to produce a reliable answer to that particular question (or questions). Thus, a research method includes not only a description of the way in which the empirical data are selected and analysed. It also includes others elements that are necessary to set up and answer a research question. For instance, the development of literature reviews aimed at clarifying the relevance of the question(s) and

[10] In the section 2.4.3 of the second chapter I discuss why I consider online teacher education as an emerging research trend.

informing the selection of theoretical and methodological tools to be applied in the research. Another component of the research method should be the *structure* that is designed in order to produce empirical evidence that could be suitable to answer the established research question(s) in a reliable way.

A research method is neither a fixed nor a ready-made recipe. The configuration of a research method largely depends on the nature of the research question. However, I think there are some steps that should be applied to any research question regardless of its nature. Here I particularly refer to the discussion/clarification of the key terms that are part of a research question. The discussion of the key terms included in a research question helps to clarify of your work the nature and scope of the research question that you are addressing to the reader. But the discussion of the key terms also helps you to guide the search for appropriate theoretical tools to study the research question. Let me illustrate these ideas by analysing the key terms of the research question 1.

1.3.1 Key terms of the first research question

A key term included in research question 1 is *online interactions*. I am aware that the term *online interaction* is commonly used to refer to any type of Internet-based interaction (synchronous or asynchronous) that takes place on the Web. However, in the research question I use the term *online interactions* to refer to the asynchronous interactions among teachers that can take place during an online course. One of the main features of the asynchronous interactions is that they are carried out through the exchange of written messages, usually posted in discussion forums. In this type of asynchronous interactions the feedback or responses to your written messages and comments are not received immediately. You can post a question in a discussion forum and get an answer some hours or even days later. The asynchronous interactions usually last several days, allowing the participants to have more time to formulate their opinions and to consider the comments and opinions expressed by the other participants. It is even possible to consult sources that are external to the online working space in order to enrich an asynchronous discussion. In section 5.2 the reader will find a more detailed description of the nature and appearance of the discussion forums and the online asynchronous discussions.

Another key term included in research question 1 is *mathematics teachers' reflections*. In this research the concept of reflection is interpreted as a mental process by which our actions, values, knowledge or feelings are consciously considered and examined. A process of reflection involves a kind of "Aha! moment" in which something is discovered or revealed. *Mathematics teachers' reflections* are those reflections that are relevant to mathematics teachers' professional development. In my research I have identified three types of mathematics teachers' reflections: *didactical reflections*, where teachers consciously consider their values and actions related to their teaching practice and the learning processes of their students; *mathematical reflections* in which teachers review their mathematical skills, knowledge and conceptions; and *extra-mathematical reflections*, where teachers consider the role and application of mathematics in non-mathematical contexts; such as its role as a gatekeeper in the education system or its application to address socially relevant issues.

In the first research question I focus on studying the asynchronous interactions that *promote* the emergence of teachers' reflections. When a teacher's reflection appears or is embedded within an asynchronous interaction, then the interaction is considered as a special kind of interaction. It is considered as an interaction that *promotes* the emergence of the reflection. This is the kind of online interactions that I am addressing in the first research question. My aim is to characterise such kind of interactions.

When I use of the expression *emergence of a reflection*, but particularly when I use the word *emergence*, I am implicitly expressing an assumption. The assumption is that I don't think reflections occur instantly. Reflections appear as a result of a process in which our actions, ideas or feelings are consciously considered. This conception of reflection as a process will be reflected in the data analysis. I will not only focus on identifying when a reflection appears within an interaction. I will also try to describe the process that gave rise to such reflection in the first place.

1.3.2 Theoretical concepts to address the first research question

After discussing the key terms of the first research question, it was clear to me that I would need at least two theoretical tools to address it. Firstly, it was necessary to find a theoretical tool to characterise asynchronous interactions. Secondly, it was necessary to establish an explicit definition of the concept of reflection. This definition should allow me to identify a reflection in an online setting.

In order to choose a tool to characterise online interactions I had at least three choices. One option was to search the literature in the area of computer-supported collaborative learning for a tool to characterise online interactions. This option would require additional time and effort to explore and get familiar with a literature that is completely new to me.

The second option was to try to find a theoretical tool produced within the area of online mathematics teacher education, which could allow me to characterise online interactions. However, as discussed in chapter 2, the online mathematics teacher education is an emerging research area. The tools for characterising online interactions do not abound in this area. The only tool I found was the analytical framework to characterise modes of participation in an online interaction used in Llinares & Valls (2009). But the framework was disregarded for two reasons. Firstly, the tool seems to produce a very general characterisation that focuses on measuring *how much* the participants interact within an online interaction, but it does not produce accurate information on *how* they interact. In addition, based on the presentation of the framework provided in the article, it is difficult to interpret how to reproduce it in a different context. The authors mention the categories of the analytical framework (provide information, clarify, amplify, etc.), but they are not illustrated with empirical data. This makes the framework obscure and difficult to replicate.

The third option was to try to find a tool to characterise interactions produced in the field of mathematics education research, and then try to adapt it to the online setting where this research was developed. This was the path I followed. I was particularly interested in locating a theoretical tool in which the concepts of interaction and reflection were related.

The tool I chose to characterise interactions was the *Inquiry Co-operation Model* (IC-Model) presented in Alrø & Skovsmose (2002). The IC-Model is a tool for characterising, from a communicative perspective, the type of interactions that occur when a group of people are faced with open–ended mathematical tasks. After studying the IC-Model, it made sense to me to use it as a tool for identifying the possible relationships between interactions and reflections. The model is based on the assumption that reflections arise from interpersonal interactions, which is a perspective consistent with my perception of the concept of reflection. The IC-Model encompasses the features that an interaction should possess in order to serve as a basis for the emergence of a reflection. So I decided to apply the IC-Model and observe which of those characteristics were present in the online interactions favouring the emergence of teachers' reflections.

I had some doubts about the feasibility of implementing the IC-Model in an online setting, because the model was developed based on empirical observations of face-to-face interactions. I did some tryouts in order to test the applicability of the IC-Model in an online setting. The tryouts consisted of applying the IC-Model in the analysis of some of the online interactions among teachers that took place in one of the online courses which I designed and applied in Mexico before starting this PhD project. The tryouts were reported in Sánchez (2008) and Sánchez (2010, b). Through them I discovered that it was actually possible to apply the IC-Model in an online setting. The way I applied this model in the research, and the potentialities and restraints I identified during its implementation are reported in the chapter 5 of the dissertation.

Establishing a definition of reflection was another aspect of the research method in which theory and practice were intertwined. Before starting the research I had an implicit, intuitive and practice-based idea of what a reflection was. During the initial stage of my research, I developed a literature review that allowed me to understand how the concept of reflection is defined in the mathematics teacher education literature. My research forced me to make explicit my own definition of the concept of reflection, but the literature review helped me to understand the differences and similarities between my own definition of the concept and the definitions provided by other researchers. After putting forward an initial definition of the concept, I started to apply it during the analysis of the empirical data generated in one of the courses that I designed (see section 1.3.3 for a discussion of the research design). The application of the concept made me aware of the need to return to my theoretical toolkit and refine the concept of reflection. In particular I found it necessary to define types of teachers' reflections. Thus, I see the configuration of the concept of reflection in my own research as the result of an interaction between theory and practice. In chapter 3 the above-mentioned review of the concept of reflection in mathematics teacher education research is presented. In the same chapter my definition of the concept of reflection is discussed in detail.

1.3.3 Research design to address the first research question

I interpret *research design* as the structure created for the purpose of producing empirical data that permit to answer in a reliable way a particular research question.

My research design is based on the development and application of two online courses for in-service mathematics teachers. The first course served to answer the first research question. This course was an introduction to the teaching and learning of mathematical modelling. The second research question (not introduced yet) was addressed with the data generated during the second course. The second course addressed the issue of the use of technology in mathematics teaching. Both courses were applied in the CICATA program as part of the subjects that the mathematics teachers enrolled in the program should pass in order to obtain a master's degree in mathematics education. During the design process of the two courses there was always a tension between the scientific and the didactical aim of the courses present. The *scientific aim* refers to the functions that the online course is aiming to fulfil within the research design of the investigation. The *didactical aim* refers to the role that the online course plays in the professional development process that the CICATA program should provide to the mathematics teachers enrolled in the program. The tension was generated when trying to design courses that could fulfil their scientific aim without detriment to their didactical aim.

In order to answer the first research question it was necessary to have access to the entities involved in the research question. I mean, it was needed to have access to teachers' reflections that were embedded in online interactions in order to study their characteristics. Therefore, this part of the research design consisted of a designing process of an online course that would promote interaction among teachers, but also support the development of teachers' reflections. The *scientific aim* of this first course was to provide me with instances of online interactions that promoted the emergence of teachers' reflections.

The decisions I made about the design of the course were based on a mixture of research results and practical experience. For example, to try to promote teachers' reflections I applied general recommendations included in the research papers I reviewed[11], such as:

1. To provide teachers with time to reflect.
2. To promote written communication.

Points 1 and 2 were covered by using asynchronous discussion forums as the primary means of communication between the participants of the online course. As part of the activities of the course I included lengthy collective discussions (five or six days long) in asynchronous forums. Such

[11] As part of my research, I did a literature review on the concept of reflection in mathematics teacher education research. In the review I focused on identifying the conditions that favour the emergence of reflections. Some of the conditions or recommendations that I found in the reviewed articles were applied in the design of the courses. The conditions that seem to favour the emergence of teachers' reflections are reported in chapter 3, section 3.3.4.

forums provided teachers with time to discuss, to plan their own comments and to analyse the comments of their peers. The asynchronous forum also forced teachers to express their ideas in a written fashion. Another good reason for including discussion forums as part of the design is that this kind of forums have been identified as a suitable space to promote mathematics teachers' reflections (see for instance Viseu & Ponte, 2009; and McDuffie & Slavit, 2003).

Another recommendation to promote teachers' reflection that was located through the literature review was to ask teachers to read mathematics education publications. According to Stockero (2008, p. 391) this kind of literature expose teachers to "alternative ideas that allowed them to begin to think about learning mathematics in ways other than how they had learned as students". Taking into account this recommendation the course included the compulsory reading of a research paper related to the content addressed in the course.

There were strategies to try to promote teachers' reflections that were based only on my practical experience. In particular I refer to the didactical device called note of reflection (see Sánchez, 2008). A note of reflection is a written case in which an imaginary situation is described. Before starting this research I had used notes of reflection as a way of introducing specific teaching situations that allow me to focus a discussion on issues that I consider relevant to address. An example of a note of reflection is used in the activity called "the marginalization index" (see chapter 4, section 4.2.3). In this activity I used a note of reflection to introduce a fictitious dialogue between a teacher and her students. The participants in the fictitious dialogue analysed some of the social and economic consequences of the application of a mathematical model used by the Mexican federal government[12]. Here the note of reflection helps to focus teachers' attention in social applications of mathematics and its consequences. I assumed that by showing to the teachers this kind of application of mathematics, I would help them to reflect on the foundation of the model and on the social and economic consequences of such application.

To try to promote interactions I also followed recommendations included in the reviewed research papers. For example, during the first course I organised heterogeneous working groups, since this is a feature that seems to promote dialogue and interactions in online settings (see McGraw et al., 2007, and de Vries, Lund & Baker, 2002). When I use the expression *heterogeneous groups* I mean that the group members have different opinions on the topic that they are analysing. To achieve this I used mathematical activities that I had previously applied in other courses for teachers. In the first course I implemented an activity called "graphs representing movements" (discussed in the section 4.2.1 of the fourth chapter), which usually produce different responses among mathematics teachers. To try to promote interaction in the course I also included an open-ended mathematical task called "the paper airplane problem" (see chapter 4, section 4.2.2). This is a modelling task that allows multiple possible solutions. I assumed that this characteristic would encourage a diversity of opinions on how to solve the task, and therefore would encourage the discussion of these ideas and the interaction.

The *didactical aim* of the course was to introduce mathematical modelling to teachers as a component of the teaching of mathematics.

[12] The mathematical model that is discussed in the note of reflection is not fictitious though. The mathematical model has been used by the Mexican federal government to determine the location of the poorest municipalities in Mexico. See Sánchez (2009b) for a discussion of this mathematical model.

Particularly, it was intended to illustrate and discuss, some of the arguments that have been provided to include mathematical modelling in the curriculum. In chapter 4 the particular activities included in this course are discussed in more detail.

In chapter 5, part of the data obtained after applying the course is presented. The data consist mainly of the comments that teachers issued in the asynchronous discussion forums developed during the course. The data were analysed using the concept of reflection and the IC-Model. Overall, the application of these two concepts consisted of the following steps: Firstly, the asynchronous discussions were studied in order to get familiar with their contents. Then the definition of reflection was used to locate instances of reflections within the asynchronous discussions. Here the definition of reflection allowed me to distinguish what was a reflection from what was not. Then the IC-Model was applied to characterise the interaction surrounding a located reflection. In other words, the IC-Model was used to characterise those interactions in which a reflection was embedded, since they were considered as interactions that promote the emergence reflections (see figure 2).

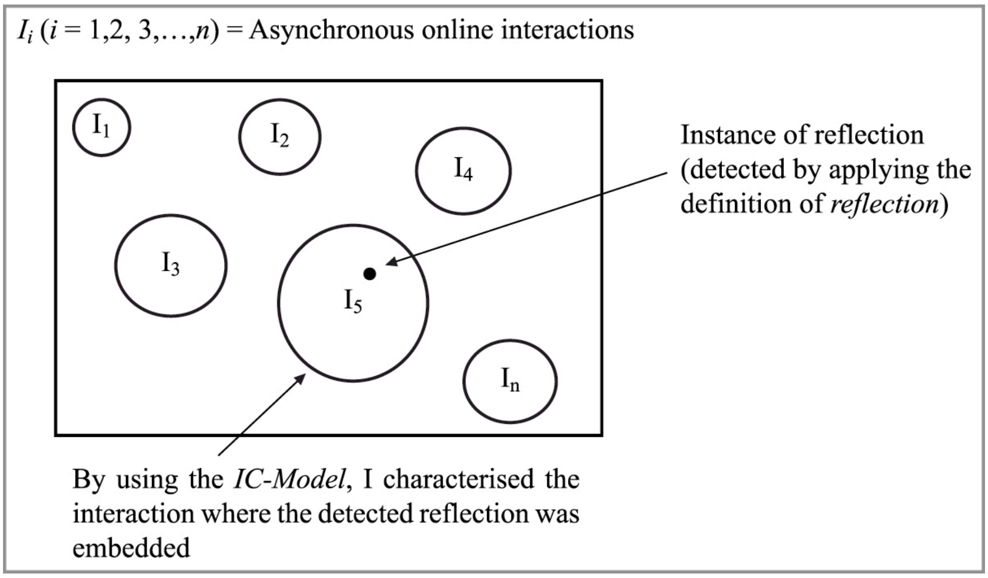

Figure 2. This figure represents the way the concept of reflection and the IC-Model were used in order to answer the first research question. First, using the definition of reflection I looked into the interactions in search of instances of reflections. When an interaction containing an instance of reflection was detected, the interaction was characterised using the IC-Model.

Finally, I tried to identify the common characteristics among the interactions that promoted reflections. I considered such common characteristics as factors favouring the emergence of reflections.

Chapter 5 presents a more detailed account of the application of the concept of reflection and the IC-Model in the data analysis.

1.4 The second research question

The second research question arose after having answered the first research question. The answer to the first research question allowed me to identify some of the communicative characteristics of the online interactions that seem to influence the emergence of teachers' reflections. However, when I was analysing the data produced during the implementation of the first course, I found empirical evidence suggesting that some non-human elements may be also influencing teachers' actions and way of thinking. One example is the use of the Excel software. The graphical and numerical information obtained through the manipulation of the software seemed to influence the way in which a mathematical problem was conceived or addressed. Before starting this research project I also witnessed situations where additional sources of information (such as web pages, activities and books) influenced teachers' views and way of thinking. After observing such situations, I assumed that some of the non-human elements included in the design of an online course had the potential to influence teachers' reflections. But what kind of non-human elements can influence teachers' reflections and how is such influence exerted? Thus, the second research question I formulated was:

(2) Which non-human elements of an online course promote the emergence of mathematics teachers' reflections?

The aim of the research question 2 is to try to identify the elements in the design of an online course that have the potential to encourage the emergence of teachers' reflections. Previous research has identified some features of online settings that promote teachers' reflections such as the act of writing or the use of discussion forums (see Ponte & Santos, 2005; Viseu & Ponte, 2009; McDuffie & Slavit, 2003; and Llinares & Valls, 2010). An answer to the second research question will contribute to the identification of features of an online setting that favour the emergence of teachers' reflections. The scientific relevance of the question lies in the identification of these features. Furthermore, an innovative methodological framework is used to trace the connections between the non-human elements of an online course and the emergence of teachers' reflections. This methodological framework can be reproduced by other researchers interested in studying the relationships between the structure of an online didactical design and the emergence of teachers' reflections.

1.5 Research method (Part 2)

In this section, the structure that was used to answer the research question 2 will be discussed. There is a parallelism between the two research structures that were designed in order to answer the research questions. In both of them I started by clarifying the key terms involved in the research question. Then I started to look for theoretical constructs that could allow me to address the question. This section will start by discussing the key terms involved in the research question 2.

1.5.1 Key terms of the second research question

The first key term that needs to be clarified is that of *non-human elements of an online course*. I perceive the structure and content of an online course as an amalgam of human elements and non-human elements. I use the term *human elements* to refer to the people who participate in an online course. In the context of my research the human elements are the mathematics teachers and the teacher educators who are participating in the courses.

Although these people are not physically present in the online course, their presence is "felt" through the ideas, opinions, questions, criticisms and feelings that they express in the messages exchanged during online interactions (in a discussion forum, in an email communication, in a videoconference). Such kind of elements are studied in the first research question. In the second research question other kind of elements are examined. When I use the term *non-human elements* I refer to the resources that a participant in an online course interact with, but which are intentionally provided by the teacher educator. These are resources that are part of the design of an online course. The resources can be of different nature: software, video, activities, articles, audio files, web pages. The two main characteristics of the non-human elements of an online course are: (1) they are elements that are intentionally provided by the course designer. The designer has control of them in the sense that he/she decides when and how they will appear within the course; and (2) they are elements that serve to represent and communicate mathematical and/or didactical ideas that are considered relevant to mathematics teachers' development. To say that a non-human element *promotes* the emergence of a reflection means that such an element has contributed to the constitution of a mathematics teacher's reflection.

It is important to clarify that in the dissertation the concepts of non-human elements, non-human components, and resources are interpreted and used equivalently.

The terms *emergence* and *mathematics teachers' reflections* are also involved in the second research question, with the same meanings that were assigned to them in the section 1.3.1.

1.5.2 Theoretical concepts to address the second research question

The same definition of the concept of reflection that was used in the first research question is applied to the second question. To address the second research question, it is necessary to find a theoretical instrument that allows to establish a connection between the emergence of a reflection and the non-human components of an online course. To try to trace such connection, a blend of theoretical concepts is used. On the one hand, I borrow the concepts of instrumentation process and instrumentalization process from the documentational approach (see Gueudet & Trouche, 2009). On the other hand, inspired by the concept of instrumental orchestration (Trouche, 2004), I have developed the concept of documentational orchestration (Sánchez, 2010a; Sánchez, to appear, b).

A *documentational orchestration* can be interpreted as the selection and arrangement of resources that a teacher educator (or a group of teacher educators) carry out with the intention of promoting the development of teachers' professional knowledge. The concept of documentational orchestration helped me to conceptualise the design of a course as a deliberated arrangement of resources. It also helped me to organise and make explicit the functions and characteristics of the resources of a course.

The concepts instrumentation and instrumentalization processes (Trouche, 2004) were used to conceptualise the way in which teachers use the provided resources, and how such resources may influence teachers' way of thinking and acting. Particularly the concept of instrumentation process allowed me to establish a connection between the reflections identified in the data and the resources that triggered them. In chapters 6 and 7 the way in which these theoretical concepts were applied is illustrated in a more detailed way.

1.5.3 Research design to address the second research question

In order to answer the second research question I designed a new online course. The second online course addressed the issue of the use of technology in mathematics teaching. The *didactical aim* of the course was to make teachers aware of the potential changes that may occur in the mathematics classroom when the use of technology is introduced. The *scientific aim* of this course was to help me to study the influence of the non-human components of the online course on the emergence of mathematics teachers' reflections.

In order to fulfil the scientific aim of the course, it was necessary to try to produce reflections in mathematics teachers. To accomplish this, some general recommendations derived from research on mathematics teachers' reflections were applied. In particular, discussion forums were used again as the primary means of interaction. As mentioned in section 1.3.3, asynchronous forums provide teachers with time to analyse the content of the discussions and force them to express their ideas and comments in a written format.

In a more particular level, different activities for the teachers were developed. One activity was aimed at allowing teachers to experience the use of mathematical techniques based on the use of mathematical software. At a later stage a *note of reflection* was used to make teachers compare the advantages and disadvantages between mathematical techniques based on the use of technology and mathematical techniques based on the use of pencil and paper. There was another activity called "analysing the pertinence of a mathematics lesson plan". In this activity teachers had to discuss how a traditional mathematics lesson plan (based on the use of pencil and paper) could be affected if applied in a classroom with ICT[13] resources. All these activities were aimed at producing mathematical and didactical teachers' reflections. Particularly reflections related to the impact that the use of technology can have on the mathematical tasks and techniques that are studied in a mathematics classroom.

The strategy to try to detect connections between the emergence of reflections and the non-human elements of the course was the following:

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[13] ICT stands for *Information and Communication Technology*. I use this term to refer to technological devices that can be used in teaching and learning of mathematics such as software, computers, the Internet, sensors, etc.

To order the set of non-human elements of the course, the concept of documentational orchestration was used. A *documentational orchestration* is an arrangement of resources that is organised in stages. Each stage has a particular purpose and comprises a particular subset of resources. Such purpose should contribute to the overall purpose of the orchestration. Thus, the concept of documentational orchestration forced me to explicitly consider and define: the resources that each stage contains, and their function and location within the orchestration (see the figure 16 included in the section 6.1 of the sixth chapter, where a graphical representation of the orchestration is provided). The documentational orchestration not only required me to structure the non-human resources used in the design of the course. It also required me to make a sort of a priori analysis of the type of "effects" that each stage of the orchestration was expected to produce.

The concept of *reflection* was applied to identify and order teachers' reflections. The concept allowed me to distinguish instances of reflections within teachers' asynchronous discussions. Furthermore, the concept allowed me to sort reflections out according to their type: mathematical, extra-mathematical or didactical.

After ordering the two sets, I focused on observing the *instrumentation and instrumentalization processes* that appeared between these two sets. That is, it was studied how teachers used the resources (instrumentalization processes), but the kinds of effects that the resources produced on teachers (instrumentation processes) were also observed. When the effect produced by an instrumentation process was a reflection, then the development and origin of such process was analysed in order to identify the particular resource which produced it. This process can be represented by figure 3.

The process consists of tracing the thick arrow back to its origin, which represents an instrumentation process generating a reflection.

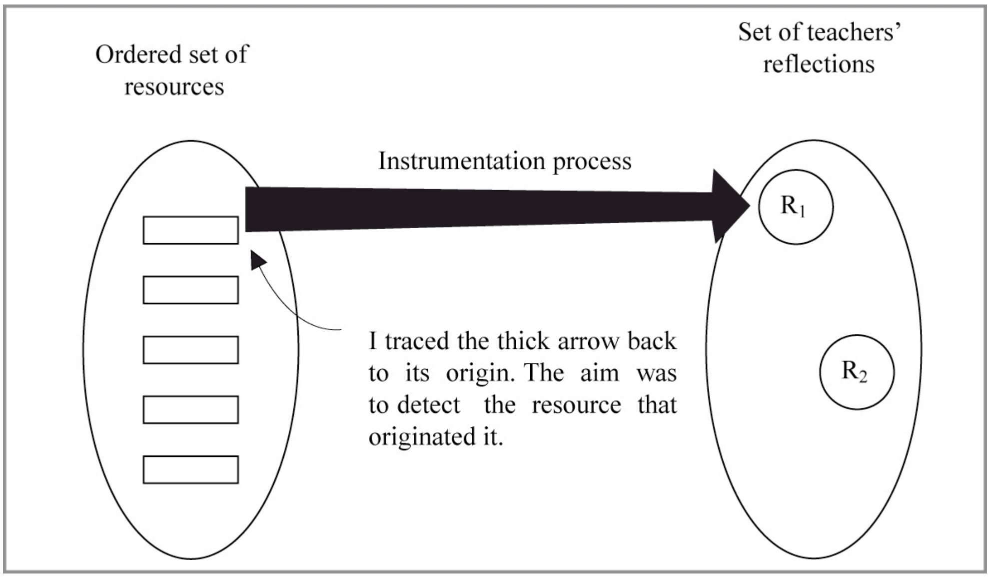

Figure 3. This figure represents the way I tried to establish connections between the emergence of a reflection and a non-human resource. When I identified an instance of reflection, I reconstructed the process that produced it in order to find its origin. During this reconstruction process I observed if any resource had contributed to the constitution of the reflection.

In this context, to "trace back" means to reconstruct the asynchronous discussion where a reflection was detected. To ensure that a particular non-human resource had actually contributed to the constitution of teacher's reflection, I looked for instances within the discussion where the teacher explicitly made reference to a non-human resource. It was taken as a necessary condition that the teacher somehow expressed the influence that the non-human resource had exerted in her thinking.

The study of the *instrumentalization processes* in the research is aimed at providing information about the way the orchestration is used by the teachers. The study of such processes help to verify which resources were used as intended and which were not. The study of instrumentalization processes is a rich source of information useful for the redesign of a particular orchestration. Details of the application of the concepts of instrumentation and instrumentalization processes are presented in chapter 7.

1.6 About the characteristics of the empirical data

The empirical data collected for this research are asynchronous discussions and written assignments produced by teachers (individually and collectively). Although the content of some of the written assignments are discussed as part of the empirical data, the most part of the data presented in the dissertation are extracts from asynchronous discussions.

The asynchronous discussions are a chain of written messages. A string of questions and answers, or comments and reactions to the comments. The written messages (also called utterances in the dissertation) that will be presented in the dissertation have the following appearance:

[28]

Theme: Re: Team 2. "Paper and pencil technique"
From: Norma
Date: Wednesday, 26th of November 2008, 00:09

Nice to meet you Homero, how are you?

You may already know Ruffini's rule (as we call it here [in Argentina]) but with a different name. It is a shortened way of solving [polynomial] divisions having the form P=(x)/(x+-b) […] To be consistent with this course, I will not recommend you any book, I will give you a direct link to a youtube video.

A picture is worth a 1000 words, don't you think? ☺

http://es.youtube.com/watch?v=RViiUlWty8M

Norma

Several characteristics should be noted here. Firstly, the message is numbered ([28] is the number of the above presented utterance). The assigned number facilitates its rapid location and reference.

The message header is located below the message's number. The header contains three elements: the title of the message, the name of the person who posted it, and the date and hour when the message was published. The original names of the teachers who appear in the empirical data have been replaced with pseudonyms in order to keep the anonymity.

After the header comes the body of the message. The messages were originally written in Spanish. The dissertation presents translations from Spanish into English of such messages. It was intended that the translation of the messages should make sense in English; therefore it was not always possible to make literal translations of the original messages. In general I have translated the utterances so that they express as best as possible the meaning that I find in the Spanish utterances. In some cases bracketed ellipsis [...] are used to denote the omission of certain segments of text. This edition was made for the sake of brevity and to increase the readability of the data I also use bracketed ellipsis to denote the insertion of words that are implicitly included in the message, but add meaning to the message when made explicit (for example the insertion of [polynomial] in the above quoted message).

The utterances often contain attachments and/or links to resources external to the online course. Where considered relevant, the contents of those files and links will be discussed.

The dissertation presents only a small selection of excerpts from the asynchronous discussions. Except for one case[14], the instances of reflection

[14] In the section 8.2.2 of the last chapter the reason for this exception is explained.

presented in the dissertation are exactly the ones that I have identified during the data analysis.

1.7 About the bibliography and citations

The bibliography and the in-text citations are organised based on the *6th edition of the APA*[15] *Publication Manual*. I particularly used the general APA guidelines provided by the *Purdue Online Writing Lab* (see http://owl.english.purdue.edu/owl/resource/560/1/).

APA recommends to provide the *digital object identifier*[16] (DOI) of an article when is available. Therefore I included them in the bibliography. The DOI is a permanent link assigned by a publisher to an article available online. Consider for instance the following reference:

> Adler, J. (2000). Conceptualising resources as a theme for teacher education. *Journal of Mathematics Teacher Education*, *3*(3), 205 – 224. doi: 10.1023/A:1009903206236

In the PDF version of the dissertation, the reader only needs to click on the DOI number of a particular article in order to be redirected to the website where the article can be accessed. Another way to reach the site where the article is stored is to write the expression http://dx.doi.org/ in the address bar of any Internet browser, followed by the DOI number of the article. In the case of the article Adler (2000), the link should be written in the address bar as:

http://dx.doi.org/10.1023/A:1009903206236

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[15] American Psychological Association (http://www.apastyle.org/)

[16] The DOI is a unique alphanumeric string assigned to an article or to a book. DOI's are intended to be stable, long-lasting links for publications available online.

1.8 Structure of the dissertation

The dissertation is divided into eight chapters. This introduction is the first chapter of the dissertation. **Chapter 2** presents a literature review where the main research trends in mathematics education are located. This review serves to clarify where my own research is located within the mathematics teacher education field. **Chapter 3** presents a review of the concept of reflection in mathematics teacher education. The chapter discusses why reflection is considered relevant to the development of mathematics teachers. In this chapter the definition of reflection that is used in the dissertation is introduced. It is also explained how an instance of reflection will be detected in the empirical data. **Chapter 4** discusses the structure and contents of the first online course designed for the purpose of answering the research question 1. **Chapter 5** shows the results obtained after applying the first online course. In this chapter the theoretical tool IC-Model is introduced. It is also illustrated how the IC-Model is applied in the data analysis. **Chapter 6** contains a description of the structure and contents of the second online course designed. This course was used to answer the second research question. **Chapter 7** introduces the concepts of documentational genesis and documentational orchestration. The chapter illustrates how these concepts are used to analyse the empirical data obtained during the implementation of the second online course. **Chapter 8** presents the results of the research. The results include the answers to the two research questions initially asked. It also includes a reflection on the implications of these research results. Particularly, the reliability, the scope and the implications on the results are discussed. The **bibliography** is presented after the eighth chapter.

1.9 Publications related to the dissertation

During my PhD studies my conception on the activity of publishing was challenged and modified. I come from an academic culture where students in mathematics education (and sometimes their teachers) do not publish frequently. There is an implicit belief between master's and PhD students that your dissertation should be finished before starting to publish your results. Here the publication ratifies and validates the work done.

At RUC I discovered that the activity of publishing could also play an educational role. You can learn a lot from the (good) reviews that you get when you submit an article. My own research was influenced by the comments and ideas I received from reviewers who evaluated articles related to different aspects of my dissertation. Since it is possible to learn from such experiences, I think the activity of publishing should not be postponed until the end of the dissertation.

During these three years I learned that you can try to publish different aspects of your PhD research. For example, if you come across an interesting book, you may write a review of it. If you apply an instructional design and get some preliminary results, you may present them at a particular conference and published them in the proceedings. A literature review carried out as part of your research can also be a research product suitable for publication.

I also discovered that the academic journals and conferences are not the only means by which you can disseminate the academic ideas related to your dissertation. You can try to popularise and disseminate your ideas through opinion articles in newspapers for example. It is even possible to use non-traditional media like YouTube[17] to bring your ideas to non-academic consumers.

The following table shows the publications produced during my PhD studies. All the publications included in the table are related to the contents of my dissertation. I divided the publications into four categories: published, to appear, submitted and rejected. The type of publication is also specified.

STATUS	TITLE	AVAILABLE AT	TYPE OF PUBLICATION
Published			
	Sánchez, M. (2007). Reseña de "Humans-with-media and the Reorganization of Mathematical Thinking. Information and Communication Technologies, Modeling, Visualization and Experimentation" de Marcelo Borba y Mónica Villarreal. *Educación Matemática*, *19*(2), 129-132	http://bit.ly/dd333W	Book review
	Sánchez, M. (2007, December 2). Matemáticas para la formación de ciudadanos críticos. *La Jornada*.	http://bit.ly/9HmDdH	Newspaper article
	Sánchez, M. (2008). *Dialogue among in-service teachers in an internet-based mathematics education program*. Discussion document presented at the study group "TSG28: Inservice Education, Professional Life and Development of Mathematics Teachers" of the 11th International Congress on Mathematical Education (ICME).	http://bit.ly/afpBnz	Conference paper

[17] YouTube is a free video sharing website on which users can upload and share videos www.youtube.com

Sánchez. M. (2009). On the fragility of an Internet-based dialogue. *Innovación Educativa*, *9*(46), 65-73.	http://bit.ly/marios	Research paper
Sánchez. M. (2009). Uso crítico de los índices y modelos matemáticos gubernamentales en el desarrollo de profesores en servicio. *Educación Matemática*, *21*(3), 163-172.	http://j.mp/7q9uxl	Research paper
Sánchez. M. (2010, Febrary 15). Gobierno y matemáticas. *La Jornada*.	http://bit.ly/9Kgkrz	Newspaper article
Sánchez. M. (2010). On the concept of documentational orchestration. En C. Winsløw & R. Evans (Eds.), *Didactics as Design Science* (pp. 11 – 22). Copenhagen, Denmark: University of Copenhagen.	http://bit.ly/bEzaaP	PhD course paper
Sánchez, M. (2010, March 12). Gobierno y matemáticas [Video file].	http://j.mp/9nG5dC	YouTube Video
Sánchez, M. (2010, b). Internet-based dialogue: a basis for reflection in an in-service mathematics teacher education program. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education. January 28th - February 1st 2009* (pp. 954 - 963). Lyon, France: Institut National De Recherche Pédagogique.	http://j.mp/bDjcZg	Conference paper

To appear

Sánchez, M. (to appear, a). Dialogue among in-service teachers in an Internet-based mathematics education program. In N. Bednarz, D. Fiorentini & R. Huang (Eds.), *The professional development of mathematics teachers: experiences and approaches developed in different countries*. Canada: Ottawa University Press.		Book chapter

	Sánchez, M. (to appear, b) Orquestación documentacional: Herramienta para la planeación y el análisis del trabajo documentacional colectivo en línea. *Recherches en Didactique des Mathématiques.*		Research paper
	Sánchez, M. (to appear, c) ¿Qué pueden obtener los profesores de matemáticas al estudiar matemática educativa? *Revista DIDAC*		Popular article
Submitted			
	Sánchez, M. (submitted) A review of research trends in mathematics teacher education.		Research paper
Rejected			
	Lezama, J. & Sánchez, M. (2009). Un estudio del proceso de formación de profesores de matemáticas en servicio en un entorno virtual. *Revista Mexicana de Investigación Educativa.*	http://bit.ly/bdNwap	Research paper

2. A review of research trends

In this chapter I present a review of the literature in mathematics teacher education research. My focus is to identify the main topics or questions investigated in this area and the main theoretical concepts used to address these questions. This review provides the reader with an overview of the field of mathematics teacher education research and clarifies where my own research is located within this area. In addition, the review will allow me to clarify and position the contributions that I am providing to this area through my own research.

2.1 Purpose of the review

In this chapter I will present a literature review of the area of mathematics teacher education research that is extensive but not exhaustive.

The purpose of this review was twofold. On the one hand and at a personal level, this review allowed me, as newcomer to the field, to obtain a general idea of the essence of this research area and an overview of the theoretical landscape it has produced. In other words, I have tried to understand what are the main concerns or questions that are being addressed or investigated by the mathematics teacher education community, and I have also tried to identify the main theoretical concepts that have been used to address those questions.

On the other hand, in the context of my own PhD research, this review has allowed me to build a conceptual map where the main research trends of this field are represented. This map has permitted me to clarify to myself (and it will allow me to clarify to the reader) where the location of my own research is within this conceptual map. As a consequence, I will be able to identify the specific areas in this map where my research is making a scientific contribution.

2.2 Limitations of the review

This review will only provide the reader with a *general* overview of the field that only highlights the *major* trends in the field. I am overlooking some areas of the field that are relevant but not having a large representation in the field as the major trends do. I am referring to areas like: research dealing with reform processes and policy issues, curriculum-based studies, constructivism, the role of communication in promoting professional growth, the use of history in teacher development, and others.

Another limitation of my review is that it does not capture the main trends within the field regarding *empirical methods* of research. For instance, I am aware that the use of video cases is widespread; nevertheless this is not reported in the review. However, in the next chapter I will report the methodological tools or instruments used to detect or identify reflections. I studied these methodological tools in order to use this information as a source of inspiration for my own research design.

2.3 Method for developing the review

The first step before starting the review was to establish some limits. These limits are defined by three questions that guided the development of the review: *what to look for?*, *how far should I look?* and *where to look?*

What to look for? As I have already mentioned in section 2.1, my interest was to identify two elements in the consulted literature: on the one hand, the main research topics in which the mathematics teacher education community is interested; and on the other hand, the main theoretical concepts used by this research community. In order to grasp these two elements, I decided to include in the review literature concerning the development of in-service mathematics teachers (the kind of teachers involved in this study), as well as research literature related to the education of pre-service mathematics teachers.

When I use the term *research trends*, I am referring to any of the two aforementioned elements; however, there are three necessary conditions in order to qualify as a trend any topic or concept. These are the *volume condition*, the *socio-geographical condition* and the *temporal condition*. The *volume condition* refers to the number of investigations conducted on a particular issue. I considered as trend those themes that are being investigated from different theoretical angles and by several different people. The *socio-geographical* condition means that, in addition to requiring different people working on the same research topic (or using the same theoretical concept), I sought for research being developed and communicated in different regions of the world. This condition allows me to ensure that there is genuine international interest about a particular topic or concept. The last condition is called *temporal*, and it refers to a particular subject that has remained as a focus of interest within the community or has been constantly researched for at least five years. I included this condition to try to avoid including in my review ephemeral research trends or research trends under a dissolution process.

How far should I look? Because I wanted to produce a more or less extensive but also updated review, I initially opted for narrowing my search to a ten years interval. Thus, the review mainly included references published between the years 1999 and 2009. However, as I will explain later, it was difficult to keep this time period as a limit during all stages of the review. At one point I had to make adjustments to this ten years interval initially established.

The development of the review was constrained by the time restrictions of the PhD research, which is the reason why it is so important to try to define an achievable literature review. That was the main cause that made me exclude from my review literature belonging to the area of *general teacher education*. In other words, most of the articles included in this review have been published in journals, books and conference proceedings pertaining to either the area of *mathematics education* or the area of *mathematics teacher education*.

Where to look? Four layers determined the literature search. Three of them are explicitly defined while the fourth is somewhat subjective. These are the descriptions of the layers:

Layer 1: When I started the review it made sense to me to use as a support other reviews of the area of mathematics teacher education carried out before my own study. Thus, the first layer consists of literature reviews on mathematics teacher education research conducted within the last decade. In this layer I included the writings of Lerman (2001); Adler, Ball, Krainer, Lin & Novotna (2005); Llinares & Krainer (2006); Ponte & Chapman (2006); Sowder (2007) and Grevholm (2008).

Layer 2: The second layer of the review is comprised of books specialised in the area of mathematics teacher education, and articles belonging to the area of mathematics teacher education published in proceedings of international conferences.

The specialized books included in this layer were: Jaworski, Wood & Dawson (1999); Lin & Cooney (2001); Strässer, Brandell, Grevholm & Helenius (2004); the four volumes of the *International Handbook of*

Mathematics Teacher Education (Sullivan & Wood (2008), Tirosh & Wood (2008), Krainer & Wood (2008) and Jaworski & Wood (2008)); Even & Ball (2009) and Clarke, Grevholm & Millman (2009).

The international conferences included in the review were the *International Congress on Mathematical Education* (proceedings from ICME-9 and ICME-10), the *Conference of European Research in Mathematics Education* (proceedings from CERME 1 to CERME 5) and the proceedings of the *Symposium on the occasion of the 100th anniversary of ICMI in Rome* (Menghini, Furinghetti, Giacardi & Arzarello, 2008). I am aware that there are other international conferences in mathematics education like for example RELME (www.clame.org.mx/relme.htm), IACME/CIAEM (www.furb.br/ciaem) and PME-NA (www.pmena.org) in the American continent. However, although such conferences are indeed international, they maintain a certain regional character and I was interested in studying the work done at conferences having a more global spirit.

My original intention was also to include in the review the proceedings of the PME conference (http://igpme.org). Unfortunately, I was unable to access them during the time I conducted this review. This shortage, however, may be considered to some extent mitigated, since the reviews of Llinares & Krainer (2006) and Ponte & Chapman (2006) included in layer 1, are based on reviews of papers published in the PME proceedings.

In the case of the CERME proceedings, I mainly focused on reviewing the reports of the mathematics teacher education working groups. These reports provided me with an overview of the topics discussed at the working group for each conference. In the case of the ICME proceedings I used the same criterion, however, I also included the individual writings (individual papers, plenary lectures) addressing topics related to mathematics teacher education. As for the proceedings of the *Symposium* *on the occasion of the 100th anniversary of ICMI in Rome* (Menghini, Furinghetti, Giacardi & Arzarello, 2008), I only included the paper of Grevholm & Ball (2008), but I also consulted some of the papers of the working group "WG2 The professional formation of teachers" of the same Symposium, retrieved from: http://bit.ly/aFlJeb

Hence, other articles included in this second layer were: from the CERME proceedings: Krainer & Goffree (1999); Furinghetti, Grevholm, & Krainer (2002); Grevholm, Even, Szendrei & Carrillo (2004); Jaworski, Serrazina, Koop & Krainer (2004); Carrillo, Even, Rowland & Serrazina (2006) and Carrillo, Santos, Bills & Marchive (2007).

From the ICME 9 proceedings I incorporated the articles of: D'Ambrosio (2004); Grevholm (2004b); Khoh (2004); Laborde (2004); Mtetwa (2004); Park (2004); Taylor & Sinclair (2004) and Sullivan et al. (2004). From the ICME 10 proceedings the following articles were included: Anthony, Graven, Grevholm & Fujii (2008); Bednardz (2008); Garuti (2008); Hejny, Jaworski, Dawson & Shiqui (2008); Llinares (2008); Margolinas, Woodrow, Cooney, Laine & Pi-Jen (2008); Park & Shin (2008), Szendrei (2008) and Vithal (2008).

Layer 3: The third layer consists of two research journals: *Educational Studies in Mathematics* (ESM) and the *Journal of Mathematics Teacher Education* (JMTE). While the two previous layers allowed me to sketch a "skeleton" of the area of mathematics education research, the third layer provided me with "the meat" or the particular pieces of research that portrayed a more clear and defined picture of the field.

I decided to include the ESM journal because I consider it one of the most important journals in the field of mathematics education research. I was interested in identifying the mathematics teacher education research that had been published in this journal. The inclusion of JMTE was an obvious choice. JMTE is currently the only specialised journal in the area of mathematics teacher education research.

When I tried to apply the ten-year limit to the third layer, I realised that the number of articles to read would be very large and therefore it would be impracticable to go through such amount of papers. So, I decided to reduce the time interval to five years. Thus, in this layer of the review I included articles published in ESM and JMTE during the period 2005-2009. In the case of ESM I mainly included papers related to the area of mathematics teacher education.

Layer 4: The fourth layer is a bit subjective because it is not focused on a particular type of publication nor limited by a well-defined time interval. The fourth layer refers to all those articles I was familiar with before starting the review, but that were relevant to inform and to shape the review. It also includes those articles that I met through the interaction with fellow researchers during the development of the review. Some of them provided me with bibliographical suggestions that were very important for the progress of the review. Other papers included in this layer were located by going through the reference lists of the papers reviewed in the previous layers.

Table 1 shows an overview of the sources consulted during the development of the review.

Layer 1	*Previous reviews (6 papers in all)*	Lerman (2001)	1
		Adler, Ball, Krainer, Lin & Novotna (2005)	1
		Llinares & Krainer (2006)	1
		Ponte & Chapman (2006)	1
		Sowder (2007)	1
		Grevholm (2008)	1
Layer 2	*Specialized books (9 books in all)*	Jaworski, Wood & Dawson (1999)	1
		Lin & Cooney (2001)	1
		Strässer, Brandell, Grevholm & Helenius (2004)	1
		The International Handbook of Mathematics Teacher Education (Sullivan & Wood (2008), Tirosh & Wood (2008), Krainer & Wood (2008) and Jaworski & Wood (2008))	4
		Even & Ball (2009)	1
		Clarke, Grevholm & Millman (2009)	1
	ICME proceedings (17 papers in all)	ICME 9	8
		ICME 10	9
	CERME proceedings (10 papers in all)	CERME 1	1
		CERME 2	1

		CERME 3	4
		CERME 4	3
		CERME 5	1
	Symposium on the Occasion of the 100th Anniversary of ICMI (Proceedings and online papers, 3 in all)	Proceedings	1
		Online papers	2
Layer 3	*Educational Studies in Mathematics (12 papers in all)*	2005 – 2009, From volume 58 to volume 72	12
	Journal of Mathematics Teacher Education (143 papers in all)	2005 – 2009 From volume 8 to volume 12	143
Layer 4	*Miscellaneous papers (38 papers in all)*		38

Table 1. Consulted sources for developing the review.

2.4 Results of the review

In this section I will present the results of the review. I will divide them into three categories: 1) *research concerns*, i.e. what are the questions or areas of interest that researchers in mathematics teacher education are currently investigating; 2) *theoretical concepts*, which are the theoretical concepts that are most used in the research field; and 3) *new trends*, which are emerging research areas that were identified in the literature review.

2.4.1 Research concerns

Teachers' beliefs, views and conceptions. Undoubtedly this is one of the most popular research areas in mathematics teacher education. Probably the interest of the community in investigating mathematics teachers' beliefs and conceptions is associated with the prevailing idea that teachers' beliefs and conceptions inform and define their teaching practices (Skott, 2009). This could explain why there is a great interest in identifying teachers' beliefs, conceptions and views about different aspects of their teaching. This could also be the origin of the effort made by some researchers to modify and develop these entities in order to positively impact teaching practice (see for example Lavy & Shriki, 2008; Grootenboer, 2008; Potari & Georgiadou-Kabouridis, 2009).

The interest in this research area has not decreased over the ten-year period covered by the review; on the contrary, researchers' interests in this area have become more specialised and their research reports and studies reflect this specialisation: we can find studies related to teachers' beliefs about their role as mathematics teacher (Lloyd, 2005); beliefs about the concept of computational estimation (Alajmi, 2009); beliefs about gender and the use of computers for mathematical learning (Forgasz, 2006); beliefs about a new educational reform (Gooya, 2007), teachers' views of mathematics (Sterenberg, 2008; Kaasila, Hannula, Laine & Pehkonen, 2008), etc.

Although research on teachers' beliefs may seem very diverse, there are prevailing trends. According to Philipp (2007), research on mathematics teachers' beliefs is focused upon: (1) understanding teachers' beliefs; (2) investigating the relationship between teachers' beliefs and practices; and (3) changing teachers' beliefs (p. 306).

Teachers' practices. This is another dominant research area in mathematics teacher education. Primarily, researchers in this area are trying to characterise the actions that the teacher performs *within* the classroom, and understand what are the factors shaping and promoting their development. In my opinion, the interest in this aspect of teachers' professional life is due to the fact that many researchers in the community believe that the most prominent part of teachers' professional work is done in classrooms (see for example Krainer & Gofree, 1999, p. 294). These kind of studies report different aspects of teaching practice *within* the classroom, for example, how teachers make real-world connections in their classrooms (Gainsburg, 2008); the types of questions asked during their lessons (Sahin & Kulm, 2008); the way teachers manage their time during a particular lesson (Assude, 2005); teachers' role in promoting collaboration among a heterogeneous group of students (Staples, 2008) or teachers' choice of examples in the classroom (Zodik & Zaslavsky, 2008).

It is important to note that a small group of researchers has begun to focus on the work done by mathematics teachers *outside* the classroom. They are particularly focused in the kind of resources used by teachers in order to define the content of their lessons or develop themselves as educators. The argument for focusing on the interaction between a teacher and the external resources she uses to plan her lessons is that this type of activity is at the core of a teacher's professional activity and development (see Gueudet & Trouche, 2009, p. 199). Another example of this type of work is Nicol & Crespo (2006). In their research they analyze how elementary pre-service teachers interpret and use curriculum materials (particularly textbooks) in their lesson planning. These researchers suggest that this type of analysis provide teacher educators with opportunities to help pre-service teachers to consider the strengths and weaknesses of their particular adaptations and designs from mathematical, curricular, and pedagogical perspectives. It also provides teacher educators with opportunities to gain insight into what pre-service teachers find important and how we might help them learn to select and pose mathematical tasks that engage students mathematically (p. 352).

Teachers' knowledge and skills. At the centre of this research area the following question is found: what kind of knowledge and skills does a person need in order to be a "good" mathematics teacher? There are many studies that underline the importance of mathematical knowledge (for example Sirotic & Zaskis, 2007; Leikin & Levav-Waynberg, 2007); but there is widespread recognition that to possess mathematical knowledge is a necessary, but not a sufficient condition for being a good mathematics teacher. It is argued that other kinds of knowledge and skills are required, such as mathematical knowledge for teaching or mathematical pedagogy (Silverman & Thompson, 2008; Koirala, Davis & Johnson, 2008); knowledge of students' cognition in mathematics (Carpenter & Fennema, 1992) and attention-dependent knowledge or awareness (Ainley & Luntley, 2007; Mason 1998; Mason, 2008). Indeed, mathematics teaching is a complex job that requires very specialised knowledge and skills. I think the following quotation captures such complexity:

> "It's one thing to know that 307 minus 168 equals 139; it is another thing to be able understand why a third grader might think that 261 is the right answer. Mathematicians need to understand a problem only for themselves; math teachers need both to know the math and to know how 30 different minds might understand (or misunderstand) it. Then they need to take each mind from no getting to mastery. And they need to do this in 45 minutes or less" (Green, 2010, March 2).

There are some theoretical models that try to capture what are the necessary skills to become a proficient or competent mathematics teacher (see Kilpatrick, 2004; Niss, 2004). Among the skills covered by these models we can find the ability to collaborate with colleagues and parents concerning mathematics teaching and its conditions, and planning effective instruction and solving problems that arise during instruction.

I think the discussion about mathematics teachers' knowledge should be shaped by the context in which the teacher develops his or her work. In other words, I think there must be some basic knowledge and skills that any mathematics teacher should have, but I also believe there are other skills and abilities that are especially needed in particular contexts. Just as Adler (2000) has pointed out: What knowledge bases [are necessary] for teaching culturally and linguistically diverse learners? And for teaching across urban and rural, under-resourced schools? (p. 210).

My impression is that the current tendency is to avoid seeing the components, skills or knowledge that make up a "good mathematics teaching" as divided and disconnected elements (see for example Bergsten & Grevholm, 2005). Researchers now are thinking on the possible balances and the connections between them.

The relationship between theory and practice. The relationship between theory and practice is an academic consideration that has been present in the mathematics teacher educators' community for many years. One concern that is at the heart of this discussion is that *theoretical knowledge* (the one produced by researchers) is usually perceived as something different and disconnected from *practical knowledge* (the one that teachers acquire through their experience). Researchers are trying to show that both types of knowledge are mutually informed, but they are also trying to explain the nature of this relationship, how to support it, and what its consequences are. When doing this review, the first article I came across which addressed this aspect was Jaworski (1999). She mentions that one of the causes of the problematic relationship between theory and practice is that educational theories are seen not to take account of the conditions and constraints of learners and classrooms that affect teachers and teaching (p. 184).

It is notable that the discussion of the relationship between theory and practice has been of particular interest to the CERME community of teacher educators. In fact at the CERME 3 conference a thematic group called "Inter-Relating Theory and Practice in Mathematics Teacher Education" was organised (see Jaworski, Serrazina, Koop, & Krainer, 2004). One of the conclusions of this working group was that more collaboration between teacher educators and teachers was needed in order to strengthen the relationship between theory and practice. This collaborative trend is reflected in the special issue also entitled "Inter-Relating Theory and Practice in Mathematics Teacher Education" which was published in the *Journal of Mathematics Teacher Education* (year 2007, volume 9, number 2). In this issue the papers written by Scherer & Steinbring (2007) and Jaworski (2007) report results of research projects that were developed through a close collaboration between researchers and teachers. This type of collaborative research in which teachers are regarded as professionals investigating their own practice, is known as *action research* and challenges the assumption that knowledge is separate from and superior to practice. The production of local knowledge is seen equally important as general knowledge. (Krainer, 2006, p. 213).

It seems to me that the relationship between theory and practice will remain one of the trends in mathematics education research in the coming years for two reasons: firstly, there are different aspects of the relationship between theory and practice that can be studied, that is to say, it is a fertile area of research. For instance, as I will argue in chapter 7, it is possible (and worthwhile) to continue exploring the use of didactical theories and other products of the mathematics education research as tools for the development of teachers (see for example Even, 2003; Tsamir, 2008); or to make explicit and confront the different views about what it means to provide a research-based teacher education (see for example Grevholm, 2004a). The second reason is that the discussion on how to address the relationship between theory and practice is still alive in recent international reports (see for example Grevholm & Ball, 2008, p. 268; Even & Ball, 2009, p. 3). I interpret this as an indication that the community of teacher educators continues to be interested in seeking ways of reducing the gap between research and practice.

Reflective thinking. Under the label of *reflective thinking* I have grouped all the research that deals with teachers or teacher educators reflecting on and learning from their own practices and experiences. This kind of research has been strongly influenced by the work of Dewey (1933) and Schön (1983), and it has remained in constant development over the ten years covered by this review.

It is clear that there is general agreement in the community of mathematics teacher educators on considering reflection as a key element in the education and development of mathematics teachers (see Lerman, 2001; Llinares & Krainer, 2006; Sowder, 2007; Chapman, 2008; Schoenfeld & Kilpatrick, 2008). Nevertheless, we can also see that the meanings attributed to the concept of reflection are varied. In the literature one can find a variety of terms such as reflective thinking, reflective stance, critical reflection, joint reflection, self-reflection, etc. that refers to different nuances and meanings of the concept of reflection. As Mason & Spence (1999) have stated: "[T]he term reflection has become too broad and diffuse in meaning to carry significance in itself" (p. 153). Due to the key role that the concept of reflection plays in my own research, I have separately analysed some of the meanings attributed to the concept of reflection in the literature. The reader will find this analysis in the next chapter of the dissertation.

2.4.2 Theoretical concepts

Research on mathematics teachers has been based on a variety of theoretical concepts. No single theory or framework dominates scholarship in this area (Grevholm & Ball, 2008, p. 268). However, there are theoretical concepts with a remarkable influence on the research community. In this section I will mention these theoretical concepts, which I also classified as research trends. As shown below, these theoretical concepts are used to study some of the research concerns described in section 2.4.1.

Pedagogical content knowledge and others forms of knowledge. As I mentioned in the previous section, one of the main concerns in the mathematics teacher education community has been to identify the kind of knowledge and skills that a teacher needs to possess in order to produce "good" teaching. The categories proposed by Shulman (1986; 1987) have been useful to conceptualise the kind of knowledge that teachers require in order to do so. The categories to which I refer are *subject matter knowledge* (SMK), *pedagogical knowledge* (PK), and *pedagogical content knowledge* (PCK).

According to Ponte & Chapman (2006), the notion of PCK was introduced in the 1990s into the field (p. 469). Since then, this one and the rest of the categories proposed by Lee S. Shulman have influenced the research on mathematics teachers' knowledge. Although the categorisation proposed by Shulman has been criticised (see for example Mason, 1998, p, 224, who claims that Shulman's taxonomy is rather unstable in practice), this categorisation has stimulated the development of new theoretical concepts better suited to the mathematics teacher's reality. One example of this is the concept of *mathematical knowledge for teaching* (Ball, Thames & Phelps, 2008), which is defined as the mathematical knowledge needed to carry out the work of teaching mathematics. According to Ball, Thames & Phelps (2008), this kind of knowledge could not be captured by the categories proposed by Shulman: "[T]eaching may require a specialized form of pure subject matter knowledge—"pure" because it is not mixed with knowledge of students or pedagogy and is thus distinct from the pedagogical content knowledge identified by Shulman and his colleagues and "specialized" because it is not needed or used in settings other than mathematics teaching. This uniqueness is what makes this content knowledge special" (p. 396).

Reflection-in-action and reflection-on-action. According to my interpretation of the literature consulted in the review, the work of Donald Schön has also significantly influenced the development of mathematics teacher education research. Particularly I am referring to the concepts of *reflection-in-action* and *reflection-on-action* (Schön, 1983). Reflection-in-action refers to the kind of reflection that practitioners perform on their own practice while actively engaged in it. This kind of reflection could lead to modifications in their practice in order to meet the immediate needs of the situation. When practitioners reflect on their practice after it has occurred, a reflection-on action takes place. Through this sort of reflection they identify their decisions and their consequences. Then it is possible to explicitly consider the actions that did not work well, and to use this experience for future planning. A basic assumption behind these two concepts is that they represent mechanisms that practitioners use to develop themselves and to learn from their own working experiences.

In mathematics teacher education research, these theoretical ideas have been particularly useful in studies of teacher educators reflecting on their own practice (see for example Tzur, 2001; García, Sánchez & Escudero, 2007) and in action research or collaborative research between teachers and researchers (for example Scherer & Steinbring, 2007). The categories proposed by Schön also served as a basis for proposing new reflection categories like that of *reflection-for-action* (see Jaworski, 1998; Scherer & Steinbring, 2007). The relevance of the concept of reflection-for-action lies in the fact that it captures the kind of reflections that educators do *before* implementing a particular teaching strategy or a didactical design.

Communities of practice. A community of practice (Lave & Wenger, 1991; Wenger, 1998) can be defined as a group of people who share an interest or a profession. Through *participation* in this collective group members learn and develop. This concept has helped mathematics teachers educators to conceptualise teachers' learning as a social process: "[I]nstead of defining learning as the acquisition of knowledge of a propositional nature, learning is conceptualized as being situated in forms of co-participation in the practices of teachers (Matos, Powell, Sztajn, Ejersbø & Hovermill, 2009, p. 170).

The influence of this theoretical concept is remarkable, not only because of the high number of studies that have employed this theoretical concept over the past ten years, but also because the concept has served as a foundation for the development of new theoretical concepts such as *community of learning* and *community of inquiry* (Schoenfeld, 1996; Jaworski, 2003; Garcia, Sánchez, Escudero & Llinares, 2007). The concept of community of inquiry for example, has allowed researchers to describe a particular community ruled by a critical mode of reflective practice that favours that the practices of the community could be continually scrutinised and reconceptualised to benefit the entire community.

Given the enthusiasm with which researchers have adopted the concept of community of practice, it is likely that its influence in the area of mathematics teacher education research continues to develop and expand.

2.4.3 New trends

One of the main contributions of this review is the identification of new research trends in the area of mathematics teacher education. Most of the trends that are identified here have not been reported in any of the reviews included in the layer 1 of the review (see table 1). One possible explanation for this is that most of the identified new trends have only become visible in recent years. These are the new research trends that have been identified:

Online mathematics teacher education. Research on online mathematics teacher education is not large when compared with other areas of mathematics teacher education research; however, this sort of research has been on the rise in recent years. The first studies about this issue that I came across when doing the review are included in the proceedings of the CERME 3 conference published in 2004 (see Garuti, 2004 and Santos & Ponte, 2004). Nevertheless, it is worth noticing that during the years 2002 and 2003 some studies discussing the use of online communication tools for communication and interaction among pre-service mathematics teachers began to appear in the literature (see Weigand, 2002 and Schuck, 2003).

Thus in general we can say that from 2002, papers from different regions of the world related with the use of the Internet for the education and development of teachers, started to appear in different settings. These studies have been reported in specialised books (Borba & Villarreal, 2006; Krainer & Wood, 2008; Even & Ball, 2009), during international conferences (Sánchez & Farfán, 2005; Ponte, Fonseca, Oliveira, Oliveira & Varandas, 2006; Guin & Trouche, 2006; Garuti, 2008; Llinares, 2008; Sánchez, 2008) and in research journals (Montiel, 2005; Ponte & Santos, 2005; Ponte, Oliveira, Varandas, Oliveira & Fonseca, 2007; McGraw, Lynch, Koc, Budak & Brown, 2007; Makri & Kynigos, 2007; Dalgarno & Colgan, 2007; Llinares & Valls, 2008; Goos & Bennison, 2008; Llinares, Valls & Roig, 2008; Sánchez, 2009a; Bueno-Ravel & Gueudet, 2009, Viseu & Ponte, 2009).

I agree with the observation that the work in this area is still scarce and is in its beginning stages (Llinares & Olivero, 2008; Ponte et al., 2009); but I can also see that researchers are opening up avenues of investigation in this area that seem very promising, such as the role of online tools in the education, collaboration and development of mathematics teachers (Ponte et al., 2007; Makri & Kynigos, 2007; Borba & Gadanidis, 2008); the nature of the new forms of discourse generated in such educational settings (Sánchez & Farfán, 2005; McGraw et al., 2007; Sánchez, 2008); or the design and structure of online learning environments (Llinares, 2008; Llinares, Valls & Roig, 2008). I am pretty sure that in the coming years we will witness the development of these and other lines of research within the online mathematics teacher education.

The design and role of tasks in mathematics teacher education. One of the fundamental premises that underlie this research trend is that: "what students learn is largely defined by the tasks they are given" (Hiebert & Wearne, 1993, p. 395). This idea can be extended to other types of learners, such as mathematics teachers or even teacher educators.

According to the findings of this review, one of the first persons who started to highlight the importance of tasks in mathematics teacher education was Orit Zaslavsky (see Zaslavsky, 1995, 2005; and Zaslavsky & Leikin, 2004). It is important to note that her interest has been focused on mathematics-related tasks. The focus on mathematics-related tasks is clearly manifested in the triple special issue of the *Journal of Mathematics Teacher Education*, where Zaslavsky participated as editor and author (year 2007, volume 10, numbers 4 to 6). However, it is also important to note that although the type of tasks reported in this special issue are mathematically based tasks, it is also discussed the kind of learning they generate, which goes beyond the mathematical content.

After the special issue of JMTE, the interest on the design, form and function of tasks in teacher education has continued to increase. An evidence of this is the sections that have been devoted to this topic on the latest books on mathematics teacher education (see section 2 of Tirosh & Wood, 2008; and Ponte et al., 2009), and particularly the book *Tasks in Primary Mathematics Teacher Education* (Clarke, Grevholm & Millman, 2009), which provides us with an international overview of the types of tasks that are currently being used in the preparation of primary mathematics teachers.

What is particular about the tasks for mathematics teacher education is that they should serve to develop the different knowledge and skills necessary for teaching mathematics. The problematic point here is that, on the one hand, such skills and knowledge are of a different nature; and on the other hand, teachers' context and background heavily influence the type of knowledge they require. For instance, Clarke, Grevholm & Millman (2009), suggest that different types of task could be needed for pre-service education and for in-service education (p. 289). These characteristics make research on tasks for teacher education complex, but interesting.

From the review it is evident that tasks for teacher education has become an area of major interest for the researchers in mathematics teacher education around the world. It seems natural to expect that this kind of research continues to develop in the coming years, not only because its noticeable popularity, but also because the role of tasks is really fundamental for the education and development of mathematics teachers.

The education and development of mathematics teacher educators. The area of mathematics teacher education is concerned with the training and development of mathematics teachers. However, it is possible to perceive that at least during the last five years, there has been a persevering interest in understanding the particular type of knowledge needed by mathematics teacher educators and how they acquire and develop such knowledge.

One of the first indicators of this interest in the education of mathematics teacher educators is found in the introduction of the book *Educating for the Future* (Strässer, Brandell, Grevholm & Helenius, 2004). The editors of the book stated that there were three major issues that guided the preparation of the symposium itself and the editing process of the book. One of those issues is *the education of teachers and their educators*. In this respect, the editors formulated questions such as: Would it be rewarding to have a special education for teacher educators? How could such an education be designed and carried out? What is the difference in the knowledge of a teacher-educator at a university or teacher training college and a 'normal' teacher at school? (p. 5).

Although the number of publications related to the education and development of teacher educators is scarce, I consider it as a research trend because of the interest this topic has awoken in leading researchers in this area. This interest can be noted by simply looking at the recent publications in the area: An example is the 15th ICMI study (Even & Ball, 2009), in which the topic was included in the book in spite of the fact that none of the papers presented during the 15th ICMI Study Conference in Brazil were related to this issue. The reason for doing this is that, according to Gómez (2009), the community is starting to recognise the relevance of exploring and reflecting on mathematics teacher educators' activities and knowledge.

Another example is the *International Handbook of Mathematics Teacher Education*, where the eighteen papers that constitute the fourth volume of this handbook are related to the development and learning of mathematics teacher educators (see Jaworski & Wood, 2008). A most recent example is the report written by Grevholm & Ball (2008), who suggest that a possible future study within the ICMI organization could be focused on teacher educators: Who they are, what they do, what they know, how they learn their work (p. 274).

There are already studies in which mathematics teachers educators reflect on their own development as teacher educators (see for example Tzur, 2001; García, Sánchez & Escudero, 2007; Krainer, 2008); nevertheless, I think there is a lot more to do in this research area since many of the questions that have been made in research on mathematics teachers could be applied to teachers educators (what kind of knowledge and abilities do they need? How do they acquire such knowledge and abilities? What beliefs do they have and how do they impact their practice? What type of tasks should be used in order to support their professional development?). My prediction then is that we are in front of another fertile field of research, which in the coming years will produce a growing amount of research related to the education and development of teacher educators. We could even witness the emergence of educational programs focused on the preparation of teacher educators, like the one reported in Even (2005).

Social justice in mathematics teacher education research. It is not easy to find a definition of social justice we can all agree upon (Gates & Zevenbergen, 2009). However, when I use the term "social justice in mathematics teacher education research", I am referring to those studies that explore the approaches and the conditions that can help us to foster and develop socially just and equitable teaching practices in mathematics teachers and mathematics educators in general. These practices should aim at ensuring a more plural mathematics teaching in the classroom, where all students, regardless of their abilities, social background, religion, gender, race and other social differences, have access to a quality mathematics education. I personally believe that this kind of teaching practices should also have an impact outside the classroom, that is, mathematics teachers should be committed to training independent and critical students able to recognise and evaluate the uses of mathematics in government and in society in general, especially where mathematics is used to support and validate decisions that significantly affect the quality of life of the citizens.

I am aware that the amount of research on social justice in mathematics teacher education is not large. In fact I would like to be cautious and say that currently this topic is not a well-established trend, but there are indications that it could become one. For example, besides the articles that can be sporadically found in the literature (such as Vithal, 2003; Forgasz, 2006; Gonzalez, 2009), I have noticed the constant presence of papers related to teacher education in journals' special issues devoted to social justice and mathematics education. Here I refer particularly to the papers of Jere Confrey and Fiona Walls included in the special issue on social justice in the *Philosophy of Mathematics Education Journal* (Number 20, June 2007); and to the papers wrote by Libby Knott y Eric Gutstein included in the special issue on social justice of the *Montana Mathematics Enthusiast* (Monograph 1, January 2007).

Another factor that made me think of this topic as an emerging trend in mathematics teacher education research was the double special issue on social justice published in the *Journal of Mathematics Teacher Education* (Volume 12, numbers 3 and 6, year 2009), in which theoretical and empirical issues on research of this type are discussed. My interpretation of these facts is that the community of teacher educators (or at least part of it) begins to recognise the fundamental role played by the teacher in the implementation of a critical and socially just education. Before being able to implement this sort of education in the classroom, it is necessary that teachers become conscious and sensitive towards the importance of this type of mathematics teaching, as well as of their didactical potentialities and limitations.

I personally believe that it is important to promote this kind of teaching practices among the mathematics teachers, especially in countries like Mexico, my home country, where there are major differences and inequities in the amount and the quality of education that their inhabitants receive. I am convinced that mathematics teachers and teacher educators can play a determining role in the positive transformation of the educational conditions prevailing in such countries. I would like to acknowledge that this conviction that I am expressing here has been stimulated by my contact with theoretical ideas such as the *critical mathematics education* (Skovsmose, 1994) and the *critical competence argument* (Blum & Niss, 1991) to which I was exposed during my stay in Denmark. Such ideas inspired me to design and apply an activity for mathematics teachers connected with these theoretical ideas (see Sánchez, 2009b), which I will discuss in more detail in the fourth chapter of this dissertation (see section 4.2.3).

It is too early to determine if the studies on social justice will actually become a trend in mathematics teacher education research. I think this will largely depend on the degree of *empathy* (Ernest, 2007) that mathematics teacher educators express towards this kind of research.

2.5 Placing my own work in the research landscape

My research is located at the intersection of two research trends: *Online mathematics teacher education* and *reflective thinking*. The research focuses on trying to identify components of an online course that have the potential to encourage the emergence of reflections in mathematics teachers, and try to clarify the nature of such influence.

Although the intersection between these two research trends represents a small area in comparison with the whole area of mathematics teacher education research, investigations that have been developed within the limits of this intersection already exist. Here I am referring to Ponte &

Santos (2005) and Ponte et al. (2007)[18]. Even though the aim of these investigations is not focused on locating the elements of an online course that could encourage the emergence of reflections, some recommendations about how to support the emergence of reflections are provided. Ponte & Santos (2005) claim that the activity of writing (which is an important means of expression and communication in an online setting) is "a powerful way of reflecting" (p. 123). In the case of Ponte et al. (2007), they point out that discussion forums are a communicative space that promotes the reflective capacity of pre-service teachers.

My research will contribute to the development of the specific research defined by the intersection between *online mathematics teacher education* and *reflective thinking*. Its main contribution will be the identification of elements of an online setting that are likely to promote the emergence of mathematics teachers' reflections. In addition I will discuss the *types* of reflections that these elements produce and the way they produce them. Additionally, this research will show useful methodological procedures to establish connections between the components of an online course and the emergence of mathematics teachers' reflections. These contributions will be illustrated in the chapters 5 and 7 of this dissertation.

2.6 What I have learned after carrying out the review?

I would like to close this chapter with a brief meta-reflection on the kind of knowledge I have gained from developing this review. First of all, the review has provided me with an overview about the current state of development of the field of mathematics teacher education research. Now

[18] To my knowledge, so far these are the only two existing investigations connecting reflective thinking with online mathematics teacher education.

I can identify the main concerns or questions that this community is addressing in their research, and what are some of the theoretical concepts used to study these questions. I have also identified that there is a core of leading researchers in this area who are the main driving force behind the development and communication of new ideas and trends in this area. But maybe this is a characteristic that is not particular to the area of mathematics teacher education research.

After conducting this review I have concluded that one of the main characteristics of the field of mathematics teacher education research is the lack of consensus regarding the meaning of key theoretical concepts. It is possible to find many definitions of the concept of belief, many definitions of the concept of reflection and so on. You cannot take any concept for granted. It is even possible to find explicit statements about it. For example, Furinghetti, Grevholm & Krainer (2002) mention that one of the questions that focused the discussion in the working group called *Teacher education between theoretical issues and practical realization* at the CERME 3 conference was: "How precisely should we define (in our papers etc.) central concepts like reflection, improvement, changes, development?" (p. 266). In a more recent publication, Grevholm & Ball (2008) refer to the central concepts and constructs used in research on teachers and teacher education. In this respect they claim: "Not always are these central concepts explained or defined generally in studies where they are used" (p. 268).

In this situation, my advice to the newcomers to this research field is to identify the main theoretical concepts or ideas playing a role in their own research, and then look into the literature to understand what these concepts mean or how researchers interpret them. This will provide the newcomers with a basis of awareness that in turn will allow them to establish their own position regarding such concepts. I myself followed this procedure in the case of the concept of reflection because of the central role played by this concept in my own research. In the next chapter I will illustrate the procedure I followed to explore this concept and the results I got as a result of this exploration.

3. On the concept of reflection

In this chapter I present a literature review that focuses on the use of the concept of reflection in mathematics teacher education research. I particularly report: (1) how the concept of reflection is defined in the literature, (2) how its relevance in the development of mathematics teachers is justified, (3) the types of methodological instruments used to detect it, and (4) conditions that are reported to favour the occurrence of reflections. I use these four elements to establish my own interpretation of the concept of reflection (which is presented in the last section of the chapter), as well as to inform some methodological decisions regarding how to detect reflections in my own research.

The literature review I presented in the previous chapter made clear to me that the concept of reflection has a great significance in the area of mathematics teacher education research. Many researchers have underlined the crucial role that reflection plays in mathematics teacher education. Here we can see two examples:

> "Reflection is the ultimate key to one's professional growth as a teacher. On a local level, the question is essentially how the day's or the week's classes went, and what one might do about that. On a more global level, the question is about not just what one does, but why." (Schoenfeld & Kilpatrick, 2008, p. 348).

> "Reflection is increasingly used to support mathematics teacher education. Emerging from Dewey's (1933) work, reflection is widely accepted as an important tool to facilitate prospective teachers' learning during their education programs. It can allow prospective teachers to make explicit and examine their initial conceptions and beliefs about teaching and mathematics that provide a basis of their sense making and learning. It also allows them to articulate, examine, question, and monitor their knowledge, beliefs, and goals embodied in their practicum teaching to develop a deeper sense of teaching and themselves as teachers" (Chapman, 2008, p. 83)

Whether we are referring to an in-service teacher or a pre-service teacher, reflection is seen as a means to develop and improve teaching practice. However, although there is an agreement in the mathematics teacher education community about considering reflection as a vehicle for the development of teaching practice, several researchers have also warned us about the lack of consensus on the meanings assigned to the concept. It is common for instance to find in the literature statements such as: "A significant feature of the literature on reflective thinking is the lack of consensus about what constitutes reflective thinking" (Mewborn, 1999, p. 316) or "Researchers have gone deeper into the concept [of reflection] over time and have adopted various definitions and theoretical frameworks for reflective thinking without consensus about a common definition having been attained" (Chamoso & Cáceres, 2009, p. 199).

Aware of this situation and given the importance of the concept of reflection within my own research, I decided to carry out a literature review that could allow me to get an overview about the different interpretations and characteristics that are assigned to this concept. Thus in this chapter I will present a literature review that focuses on the use of the concept of reflection in the mathematics teacher education research literature. Before presenting the results of the review, I will explain in more detail the purpose of the review and how it was conducted.

3.1 Purpose of the review

It is important to notice that the purpose of this review is different from the review presented in chapter 2. While the previous review allowed me to situate my own work in the mathematics teacher education research landscape, the role of this review was to inform and shape some elements of the theoretical and methodological components of my research.

With regard to the theoretical elements of my research, this review allowed me to track down some of the different interpretations of the concept of reflection that exist in mathematics teacher education research. The analysis of the current state of development of the concept helped me to clarify and make explicit my own stance on the concept of reflection.

During the review the type of methodological instruments that are used to detect a reflection were identified. Also, the type of evidence that is provided to prove or to make evident its existence was located. This information served as a source of inspiration to define the type of strategies that will be used in this research in order to identify reflections in an online setting. See section 3.4.3 for a brief discussion on such methodological strategies to identify reflections.

The review also focused on identifying the types of approaches or strategies that are claimed to promote the emergence of reflections. This information was taken into consideration to inform the design of the online courses for teachers that were applied during this research.

3.2 Method for developing the review

This review can be considered as an extension of the more general review presented in the second chapter. However, in this review the research trend called *reflective thinking* was studied in greater depth, paying particular attention to the concept of *reflection*.

The review includes studies related to the concept of reflection, regardless of the educational level considered in the study (pre-service teachers, in-service teachers or even teacher educators). The articles included in the review were those that use the concept of reflection, or any related concept such as reflective thinking, reflective stance, critical reflection, joint reflection, collegial reflection, self-reflection and qualified pedagogical reflection. I considered necessary to include all these concepts in the review, because when I started looking for papers related to the concept of reflection (by looking for the word "reflection" in the titles, abstracts and keywords of the papers), all the above-mentioned concepts came up. I think this is an indication of the variety of stances on the concept of reflection that currently exist.

The four layers that defined the review presented in chapter 2 also bounded the consulted sources for this review. Nevertheless in the present review the length of the fourth layer was extended. For example, many papers considered in this review were located through the references lists of the articles included in the first three layers. Carrying out searches in educational databases (such as ERIC and MathEduc)[19] was another way in which the fourth layer was extended. Through these searches, publications included in journals not specialised in mathematics education but relevant to this study were located.

In addition to the selection of sources, the review was guided and organised by the following four key questions:

1. How the concept of reflection is defined in the literature?
2. What sorts of arguments or reasons are provided to justify the importance of reflections in the education and development of mathematics teachers?
3. What kind of instruments are used to detect reflections?

[19] *ERIC:* www.eric.ed.gov and *MathEduc:* www.zentralblatt-math.org/matheduc

4. According to the results obtained in the reviewed articles, what elements, approximations or strategies are claimed to promote mathematics teachers' reflections?

3.3 Results of the review

This section presents the answers to all the previously mentioned questions. After that, my own definition of the concept of reflection will be explicitly formulated.

3.3.1 How the concept of reflection is defined?

While trying to answer the question *how the concept of reflection is defined?* I found that at times the concept is not well defined in the research papers. Moreover, when the term is explicitly defined, the interpretations of the concept may be slightly different among papers. In an effort to organise the information found, the following categories regarding the way research papers refer to the concept of reflection were established:

A first category consists of those **papers that use the notion of reflection without defining it in the body of the document**. Within this category of papers it is common to find the word *reflection* in the title of the work, in the abstract or even as part of the keywords. The concept is also mentioned in the body of the writing through phrases like "we established a new theme to promote reflections" or "The value of the model lies in its usefulness as a guide for enabling teachers to reflect on their instructional practice", nevertheless the concept of reflection is not specified or defined. This type of papers did not provide me with information to answer the question previously raised. This category includes articles such as Artzt & Armour-Thomas (1999), and Ponte et al. (2007).

One possible explanation for the absence of explicit definitions of the concept of reflection in these articles could be that the concept of reflection is not central to these investigations. That is, we sometimes use terms such as "teaching" or "learning" without explicitly defining them in our articles, but just using them in an intuitive way. This usually happens when such concepts are not the main focus of our research. This may be the case of the articles included in this first category.

A second category is composed of those **papers that use the notion of reflection, but without explicitly defining it in the article. However, some sort of implicit characterisation is provided through the empirical data or the references used in the article**. An example of such kind of papers is Scherer & Steinbring (2007). A central concept in this work is that of joint reflection, and though not explicitly defined in the body of the paper, the concept could be characterised as a process linked to the collective analysis of teaching episodes. I affirm this because the paper reports the development of a project in which mathematics teachers and researchers collaborate. During the project, teachers plan and prepare mathematics lessons. Afterwards these lessons are applied and registered (by means of video recordings and field notes). Finally, teachers and researchers analyse together the video-recorded classroom episodes. It is also possible to notice that in the concluding remarks of the paper, the authors pointed out to the need of using the concepts of reflection-on-action, reflection-in-action and reflection-for-action in order to make distinctions between the types of reflections that occurred during their project (p. 177). This confirms my claim that the authors interpret reflection as an act connected with the analysis and inspection of the teaching practice. Other articles in this category are Zaslavsky & Leikin

(2004), Ponte & Santos (2005), Goodell (2006), Nührenbörger & Steinbring (2009) and Ryken (2009).

The third and last category consists of **papers that explicitly assume or provide a definition of reflection**. In this group I have included the articles of Jaworski (1998), Mewborn (1999), Walen & Williams (2000), Wood (2001), Krainer & Thoma (2001), Tzur (2001), Hodgen (2003), Bjuland (2004), Ticha & Hospesova (2006), García, Sánchez & Escudero (2007), Stockero (2008), Jansen & Spitzer (2009), Chamoso & Cáceres (2009) and Spanneberg (2009). Most articles in this category consider reflection as a mental process in which a teacher considers and inspects in an explicit way her own teaching practice. For instance, according to Krainer & Thoma (2001) reflection is: "The attitude towards, and competence in, systematic and critical analysis of one's own actions and work" (p. 54); Chamoso & Cáceres (2009) claim that: "Reflective thinking is a deliberate thinking about action and trying to improve it" (p. 199); in turn Spanneberg (2009) declares: "I interpret reflection in the mathematics classroom as a process whereby teachers continuously examine, plan and revise their own practice" (p. 52).

We could say that there is a consensus among the authors of the articles included in this third category, about considering reflection as an act associated with the explicit and conscious analysis of our own practices and actions. However, there are some differences in the type of skills and knowledge that teacher educators want to develop in mathematics teachers through reflection. For instance, in the work of Stockero (2008) and Jansen & Spitzer (2009), reflection is aimed at helping teachers to identify relationships between their own teaching practices and the mathematical understanding of their students: "reflection [is] defined as analyzing classroom events in order to identify often subtle differences in students' mathematical understandings and the ways in which the teacher's actions contributed to them. A reflective stance, then, is defined as the ability to reflect in this manner." (Stockero, 2008, p. 374 – 375). A different situation is the one presented by Hodgen (2003), in which the reflections produced by a teacher helped her to "begin to transform her beliefs and knowledge about school mathematics" (p. 4).

Thus, when attempting to answer the question: *How the concept of reflection is defined?* I have found that the concept of reflection is sometimes used without being clearly defined. But when researchers provide an explicit definition of the concept in their research, there is a kind of agreement about considering it as a process in which mathematics teachers examine their own practice and actions in an explicit fashion. However, there is not an agreement about the aspects of the teaching practice that is developed through this process of reflection. For example, in some cases it is said that the teacher can develop her mathematical competencies (as in Hodgen, 2003), while other researchers talk about the development of subject-didactic and pedagogical competencies (Ticha & Hospesova, 2006).

3.3.2 Arguments to justify the relevance of reflection

In her extensive literature review on the mathematical education and development of teachers, Sowder (2007) points out: "Throughout the research discussed in this chapter, the topic of reflection and its role in furthering teachers' professional development and education is a constant in the discussions. The importance of teachers' reflecting on their practice is a recurring theme in research on teaching" (p. 198). Indeed, many researchers seem to argue that reflection is central to the development of mathematics teachers, but what exactly are the reasons that are provided to support this assertion?

Reasons are diverse. The given reasons are related to the definition of reflection that is being handled, or to the type of activities or approaches used to promote it (this point is discussed in more detail in section 3.3.4). But I would say that the reasons provided are of three types:

(1) Reflection as a means of obtaining knowledge. It could be knowledge about aspects of your teaching activity, or even about your own mathematical knowledge. Consider the example of Tzur (2001). This is a self-study which attempts to conceptualise the process of development of a teacher educator. The author produces and analyses narratives of experiences that impacted his development as a teacher educator. One of these narratives is related to his student days, when he was encouraged to tutor his peers and to serve as a leader for younger students. In this respect the author comments: "[I]t seems that my continuous reflections on the activities that I used to explain my solutions to others played a key role in advancing my mathematical understanding […] A key aspect of this learning is that it was the tutor who learned mathematics via reflection on teaching activities and on learners' work in response to these activities" (p. 265). A different perspective is the one provided by Nührenbörger & Steinbring (2009) and Scherer & Steinbring (2006) where reflection is seen as a means for obtaining knowledge about the problems and conditions of the teaching activity and their relationship with the mathematical students' learning processes.

(2) Reflection as a trigger for changes. Other researchers have argued that the process of reflection is related to changes in mathematics teachers' way of thinking and beliefs. Some researchers even claim that the process of reflection could lead to actual changes in teachers' professional practice. For instance, Ticha & Hospesova (2006) present a study in which teachers and researchers collaborate to develop and implement instructional experiments. These experimental interventions were video-recorded in order to use them later for making joint reflections (here interpreted as collective analysis of the video) on their contents. When the authors report the effects that such activities had on teachers, they claim: "On the basis of our observation, we are convinced that joint reflection is an effective way of improving teachers' competence. Pedagogical competence was enriched by the insight into the thinking of individual pupils. We perceived the development of more sensitive teachers' approaches to pupils' ways of thinking and of the ability to use them in teaching. Teachers had an opportunity to become aware of the level of their mathematical knowledge and in their teaching and to remove them, i.e. they improved their subject-didactic competence" (p. 150). Jaworski (1998), Wood (2001) and Hodgen (2003) are other examples of studies conceptualising reflection as a trigger for changes in mathematics teachers.

(3) Reflection as a link between theory and practice. To argue that reflection can be a link between theory and practice is a kind of argument often used in those papers in which mathematics teachers and teacher educators/researchers are working collaboratively. There are examples where the relationship between theory and practice is addressed through reflection. Such is the case with the self-study carried out by García, Sánchez & Escudero (2007), in which the authors identify different moments (or "steps", as they called them) in their development as mathematics teacher educators. The researchers then reflect on these moments in order to develop a theoretical model to study the relationship between theory and practice. In this research, reflection plays a determining role in the study of this relationship: "We have recognised different steps in our study. Their identification and reflection on them, and the way in which they relate to each other, has made it possible for us to characterise a way of considering the relationship between theory and practice in mathematics education" (p. 14).

3.3.3 What kind of instruments are used to detect a reflection?

Another aspect that I tried to understand through the literature review was the kind of methodological tools or instruments that are used as a means for detecting a reflection. I was also interested in identifying the sort of evidence that is provided in order to argue that a reflection has taken place. Such information served me as a reference for planning and defining some methodological aspects of my own research, particularly those aspects concerning the ways of detecting a reflection in an online setting.

Regarding the tools that are used as a means for detecting a reflection, it is notable the use of written accounts produced by the mathematics teachers. The accounts are usually written reports that are based on teachers' field experiences. This is the case of Stockero (2008): "Following each field experience, the PTs [prospective teachers] were required to write a paper in which they reflected on their experience by analyzing how they as the teacher helped or hindered the development of students' mathematical understanding of the problem, as well as analyzing and interpreting the mathematical thinking of the students in their group" (Stockero, 2008, p. 378). Other examples of research in which teachers are asked to produce written reflections are Ticha & Hospesova (2006) and Jansen & Spitzer (2009).

Another widespread way to identify reflections is through the analysis of group discussions. The discussions can be organised around the analysis of the solution process of a mathematical problem (like in Bjuland, 2004). However, the most part are based on the analysis of video cases. The video cases usually contain teaching episodes performed by the same teachers who are analysing them (see for example Wood, 2001; Scherer & Steinbring, 2006; Nührenbörger & Steinbring, 2009). The role of video cases is to detonate discussions about the nature of the interactions among the teacher, her students, and the mathematical tasks involved.

Another tool for locating reflections is the use of interviews and questionnaires. Through these instruments researchers attempt to capture the experiences that teachers obtain through their field work (Artzt & Armour-Thomas, 1999) or during an in-service course (Ponte & Santos, 2005).

The type of evidence that is provided to illustrate the emergence of a reflection is closely related to the kind of instrument that was employed to register it. For instance in the work of Ticha & Hospesova (2006), both, written reflections produced by teachers and group discussions of video cases are used. Then the authors present excerpts from the transcripts of those discussions, as well as excerpts from teachers´ written productions in order to illustrate the existence and development of reflections among teachers. It is interesting to note that although the majority of the papers present a qualitative analysis of the collected data, some research also uses quantitative analysis to illustrate changes and developments in teachers' reflections (Stockero, 2008; Chamoso & Cáceres, 2009; Goodell, 2006).

3.3.4 How reflections are encouraged?

According to the reviewed studies, there are several elements that seem to influence the emergence of a reflection. Such elements are of a different nature. In some cases the elements are *activities*, and in other cases they refer to *conditions*. I have included a third category of elements called *resources* that although is comprised of only one element, it is relevant to my own research. Next each of these categories are illustrated:

Activities. Several authors claim that the *act of writing* is a vehicle for reflection (Ponte & Santos, 2005; Zaslavsky & Leikin, 2004; Tzur, 2001; Spanneberg, 2009). As Ponte & Santos (2005) assert: "[W]riting is a powerful way of reflecting, helping teachers to clarify ideas, to look at them from different angles, to come back and revise; the steadiness of the written word also seems to provide more depth to the ideas" (p. 123).

Other authors claim (although this claim is sometimes implicit) that the *analysis of video cases* facilitates reflection (see Stockero, 2008; Jansen & Spitzer, 2009; Wood, 2001; Scherer & Steinbring 2006; Nührenbörger & Steinbring 2009). According to Jansen & Spitzer (2009): "Research supporting the development of presence, or 'learning to notice' […] suggests that prospective and practicing teachers can make progress toward engaging in reflective thinking through participating in analysis of video cases" (p. 147).

Shari L. Stockero suggests that *the reading of mathematics education publications* is another activity that improves the level of reflection: "Course readings, for example, exposed the PTs [prospective teachers] to alternative ideas that allowed them to begin to think about learning mathematics in ways other than how they had learned as students.

Without these readings to draw upon, the PTs may not have had the tools necessary to reflect at higher levels" (Stockero, 2008, p. 391).

In turn Bjuland (2004) affirms that the act of *reflecting on the own solution process of a mathematical activity* promotes the reflection among pre-service teachers about their preparation for the teaching profession (p. 221).

Conditions. The relevance of *time* in the emergency and the depth of a reflection has been highlighted by several researchers (see Sowder, 2007; Chamoso & Cáceres, 2009; Jaworski, 1998; Jansen & Spitzer, 2009; Ticha & Hospesova, 2006). For instance, Sowder (2007) underlines: "[T]ime is needed for developing the ability and habit of reflection. Reflection rarely occurs when time is not a resource available to teachers"(p. 198).

Mewborn (1999) and Hodgen (2003) refer to the ability to be *distanced* or *decentered* from our own practice or actions as a condition for producing a reflection: "A precondition for the act of reflection is the ability of the person to decenter and view his or her actions as a function of the context in which he or she is acting. Schön's (1983) reflective practitioner, a notion that enjoys so much credence in the field of education, cannot exist unless the individual is willing to step out of himself or herself and view his or her actions from a relativistic perspective. (Cooney and Shealey, 1997, p. 100, quoted by Hodgen, 2003, p. 1)".

Finally, Mewborn (1999) mentions several characteristics that should be part of a teacher education environment in order to promote reflections. She refers to environment's characteristics such as:

- It must promote feelings of trust
- It must permit dissent and conflict
- It should be supportive and challenging
- It should be based on mutual respect

Resources. As mentioned at the beginning of this section, I decided to include this category because I considered it relevant to my own study. In the work of Ponte et al. (2007) some of the components of an online setting that promote reflection are discussed. One of these components is the discussion forum. In this sense the authors claim: "If the aim is to promote student teachers' reflective capacity through the development of a virtual community or a learning network, then it makes sense to stress the discussion forum" (Ponte et al., 2007, p. 87 – 88). There are other studies where it is claimed that the discussion forums are a suitable space to promote mathematics teachers' reflections. See for instance Viseu & Ponte (2009) and McDuffie & Slavit (2003).

Some of these activities, conditions and tools were taken into account during the design of the online courses for mathematics teachers that I developed. In chapters 4 and 6 the specific way in which some of these elements were used in my designs will be illustrated.

3.4 My position regarding the concept of reflection

I will end this chapter by outlining my own standpoint on the concept of reflection. Firstly, I will introduce and illustrate the definition of the concept of reflection that will be used throughout the research. Secondly, the similarities and differences between my own definition and the definitions found in the literature review will be discussed. Finally, I will discuss in general terms the methodology I will use to identify teachers' reflections in an online setting.

3.4.1 My definition of the concept of reflection

I think of reflection as a cognitive activity, a process of thinking. It is a mental process by which our actions, beliefs, knowledge or feelings are consciously considered and examined.

To reflect involves more than just recalling or considering something consciously. A process of reflection provides enlightenment about the actions or ideas that are being considered. A process of reflection involves a kind of "Aha! moment" in which something is discovered or revealed. I want to illustrate this idea with an example.

Example 1. I made this example up to try to illustrate my own interpretation of the concept of reflection. The example consists of a dialogue among two mathematics teachers. One of the teachers, named Luis, has been participating in an in-service course on mathematics education. As part of the course activities, Luis has been reading some research papers. The dialogue begins when Luis comes across a colleague who asked him about the in-service course he is attending:

Luis: Hello Julio!

Julio: Hello Luis. How are you doing? How is the course you are attending?

Luis: It has been very interesting! I have just finished my homework this morning.

Julio: Yes? What does the homework consisted of?

Luis: I had to read a paper called tacit models and infinity[20]. It is quite interesting.

Julio: Why do you think the paper is interesting?

Luis: Well, in the paper it is claimed that when we have to deal with concepts which are highly abstract or very complex, our reasoning tends to replace

[20] Here Luis is referring to the paper Fischbein (2001) which is included in the bibliography.

them by substitutes which are more familiar, more accessible, more easily manipulated. These are mental models. Sometimes, mental models are used intentionally, consciously, but sometimes we are not aware of their presence and/or of their impact. Apparently these kinds of models have a considerable effect on our thinking strategies and conclusions.

Julio: Mmmm…

Luis: And you know Julio? The paper made me remember an experience I had two weeks ago.

Julio: Yes? Tell me about it.

Luis: I was in the classroom with my students. We were studying the characteristics of the graph of the function $y = \log(x)$.

Julio: o.k

Luis: I draw on the blackboard a graph like this one [Luis shows to Julio a piece of paper with a drawing on it. See figure 4]. Then I tried to explain them that in this region, the graph approaches the y-axis but never touches it. [Here Luis is referring to the vertical asymptote of the function]

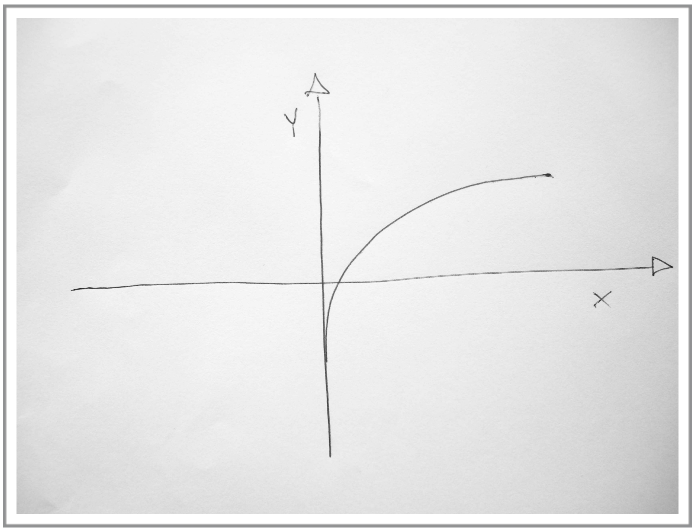

Figure 4. Representation of the graph used by Luis to illustrate the graphical behaviour of the function $y = \log(x)$.

> **Julio:** Yes
>
> **Luis:** To me it was something very natural to refer to this property, but then one of my students asked how that was possible. She said that if she walked towards one of the classroom's wall, even by using very short steps, there would be a moment when she would reach the wall. Then she asked how it was possible that the function would not touch the y-axis.
>
> **Julio:** I see.
>
> **Luis:** But to be honest Julio, I sort of ignored her. I though she was just a dull pupil unable to understand the mathematical idea I was trying to explain. Nevertheless, after reading the paper about tacit models and infinity, I realised that my student was actually using a mental model. She was using a mental model where the behaviour of the asymptote was substituted for the metaphor of "walking towards a wall". This is the reason why she found difficult to grasp the mathematical idea I was explaining. I think I should try to pay more attention to this kind of situations…

I consider the above example as a manifestation of a reflection process that Luis has experienced. He is not only recalling and consciously considering one of his teaching experiences. He also discovered an aspect within that experience that previously was not visible or perceptible. I refer to the mental model that his student probably used to try to understand the concept of asymptote.

Thus, a reflection not only consists in explicitly considering your actions, values, knowledge or feelings. A reflection also implies that an aspect of the element being considered is discovered or becomes visible. This is what I mean by the "Aha! moment".

The previously presented example illustrates a process of reflection which is anchored in a teaching experience. However, there are other kinds of reflections that are also important to the development of mathematics teachers. In particular, in my research I have identified three types of teachers' reflections: didactical reflections, mathematical reflections and extra-mathematical reflections.

A **didactical reflection** refers to the process of reflection in which a teacher consciously considers her teaching practice. Her values and actions associated with this practice and/or the consequences of such values and actions. The above-mentioned example 1 is an instance of this kind of reflections.

In a **mathematical reflection** a teacher consciously considers and revisit aspects of her own mathematical knowledge. During this type of reflection a teacher consciously examine for example, her interpretation of mathematical concepts or her way of solving mathematical tasks. Such reflections can lead to an improvement of teachers' mathematical knowledge, since this kind of reflections can help the teachers to identify personal misconceptions or even help them to acquire new mathematical knowledge.

An **extra-mathematical reflection** occurs when a teacher consciously considers the role and application of mathematics in non-mathematical contexts. It can also include a consideration of the consequences of such application. For example, by means of an extra-mathematical reflection, a teacher can become aware of the undesirable and irreversible societal and economical consequences that a mathematical-based decision-making model can potentially produce in the life of the members of a particular community.

3.4.2 Comparing my definition of the concept of reflection

A fundamental similarity between my definition of the concept of reflection and the rest of the definitions found through the literature review is that reflection is interpreted as a mental process in which *something* is considered or examined in a conscious way.

I wrote "something" using italics because researchers in mathematics teacher education usually interpret such "something" as the act of teaching. In other words, researchers in mathematics teacher education lay particular emphasis on the kind of reflections which are anchored in teaching practice. The widespread use of video recordings in reflection research, through which teachers are asked to analyse classroom episodes, can be considered as an evidence of the emphasis on reflection on teaching practice. The extensive use of theoretical concepts such as reflection-for-action, reflection-in-action and reflection-on-action is another kind of evidence of this emphasis on teaching practice.

However, in my interpretation of the concept of reflection not only the teaching practice can be the focus of a reflection. You can also reflect on your mathematical knowledge, or even on your feelings and values.

I am not claiming that my definition is novel or original. For instance, Bjuland (2004) focuses on pre-service teachers' reflections related to the analysis of their solution processes of geometry problems. I would qualify such kind of reflections as mathematical.

One difference between my definition of the concept of reflection and the other definitions found is that, in my definition, emphasis is placed on the stage of discovery or revelation (the "Aha! moment") that a reflection can produce. Such "Aha! moment" is connected with the outcomes of a reflection. More specifically, I think of a reflection as a means for getting or improving knowledge (it could be mathematical knowledge acquired through a mathematical reflection), but also as a way for becoming aware of your own ideas and values about the teaching and learning of mathematics.

3.4.3 How do I intend to detect a reflection in an online setting?

The literature review presented in this chapter has shown that one of the means most commonly used by researchers for detecting reflections is the analysis of written accounts produced by mathematics teachers. The analysis of group discussions is another method used by researchers in order to detect teachers' reflections (see section 3.3.3). These two methods for detecting reflections could be applied to an online setting.

In an online setting, as the one framing this research project, it would be a "natural" methodological decision to try to identify the emergence of reflections through the analysis of written accounts. I claim this since most part of the communication and the interaction within this sort of educational setting is carried out in a written form. Thus, I will try to identify instances of reflections through the analysis of the teachers' written productions. Here I am referring to the individual and collective reports that teachers have to produce during the courses. *I will pay particular attention to the asynchronous discussions*, which are conducted through the exchange of written messages. In the fifth chapter of the dissertation (see section 5.2) it will be explained in detail how the asynchronous discussions are analysed.

A supposition of my work is that, although a reflection in a non-physical entity, it is possible to detect it. However, I am aware that the analysis of written accounts will allow me to identify instances of teachers' reflections, but in an indirect manner. The analysis of written accounts does not allow me to access a reflection directly, but only allows me to access manifestations of such reflection. In other words, during the analysis of teachers' written productions I will look for manifestations of the outcomes or effects of a reflection. Manifestations of the "Aha!

moment". I will consider such manifestations as evidence that a reflection has occurred. For example, in the previously fictitious dialogue between the teachers Luis and Julio, I interpret last Luis's utterance as an indicator that he has experienced a reflection. When Luis says that he has discovered a possible explanation for the difficulties experienced by his student in understanding the nature of an asymptotic behaviour, I interpret this discovery as part of an "Aha! moment" he had experienced. Therefore I also interpret the discovery as evidence that Luis has undergone a reflection process.

It is worth noticing that in several of the analysed studies, *researchers explicitly ask teachers to produce reflections*. Let us take the following two quotations as examples:

> "[T]he PTs [prospective teachers] were required to write a paper in which they reflected on their experience by analyzing how they as the teacher helped or hindered the development of students' mathematical understanding of the problem" (Stockero, 2008, p. 378).

> "[T]eachers were assigned to write a first attempt at a reflection on their teaching. In this reflection, they were asked to make claims in which they described students' thinking during the lesson and support those claims with evidence" (Jansen & Spitzer, 2009, p. 138).

My point here is that my research will focus on identifying only those kinds of reflections that occur *spontaneously*. I think that if I want to study the possible relations between the elements of an online course and the emergence of teachers' reflections, then I must let those elements act for themselves, in an independently way. Then I could study their possible influence in the emergence of teachers' reflections. In the context of this research it does not make sense to explicitly ask teachers to reflect.

Avoiding asking teachers to explicitly produce reflections is a methodological decision that has implications for the empirical data that I will get for this research. One consequence is that I am expecting to detect free and authentic reflections (not “forced” reflections). Another consequence is that, very likely the number of reflections detected will not be large. That is, although the proposed online designs may be adequate to produce reflections in teachers, this does not guarantee that such reflections will be expressed or manifested in the online setting. It is possible to experience a reflection without feeling the need of sharing it or expressing it.

4. The first online course: Modelling

This chapter introduces the first online course that was designed as part of the research method of this PhD research. The course is an introduction to the teaching and learning of mathematical modelling. During the course three of the arguments in favour of the inclusion of mathematical modelling in mathematics instruction were illustrated and discussed. The scientific aim of the course was to produce online interactions that favour the emergence of teachers' reflection. The chapter shows the activities used and steps that were taken in order to meet such aim.

Throughout the two previous chapters I have tried to do two things. Firstly, to explicitly locate my own research within the theoretical landscape of mathematics teacher education research. Secondly, to provide the explicit definition of the concept of reflection that I will use in this research.

In this chapter I will refer to the first of the two courses I have designed as part of my research method. This course reflects the early stage of my research in which my attention was focused on finding the possible relations between the emergence of teachers' reflections and online human interactions. The course was designed to try to promote interaction among mathematics teachers, but also to support the emergence of teachers' reflections.

The chapter presents a description of the course. The presentation is divided into two parts. In the first part the general structure of the course (topic addressed, duration and didactical aim) is considered. In the second part the particular activities that were included in the course are discussed.

4.1 The general structure of the course

As discussed in the introduction of this dissertation, the two courses that I designed as part of my research method were applied in the CICATA program, which is an online graduate program aimed at in-service mathematics teachers working at different educational levels: primary, secondary and university levels. This graduate program is hosted by the *National Polytechnic Institute of Mexico*[21]. Despite the institutional context in which the courses were implemented, I had complete freedom to decide on the structure and content of the courses.

The first course I designed was called "Introduction to the teaching and learning of mathematical modelling". The course lasted five weeks and a value of 8 credits out of 76 was assigned to it[22]. The participants took the course during the months of March and April of the year 2008, during the second semester of their graduate studies. The course also involved the participation of a group of five teachers educators (including myself). Before the course started, I discussed with them the didactical aim of the course and the content of the activities. They knew that the empirical data generated during the course would be used in my PhD research. The main role of the teacher educators was to observe the development of the course and to participate as facilitators in some discussion forums.

The main reason why I decided to design a course where teachers could discuss aspects of the teaching and learning of mathematical modelling was my perception that there is an institutional tendency across Latin America aimed at including mathematical modelling in the mathematics curriculum at secondary level. One example of this tendency is Colombia.

[21] http://en.wikipedia.org/wiki/National_Polytechnic_Institute

[22] 76 is the total number of credits required to obtain a master's degree at this University.

In 2006 the National Ministry of Education of Colombia officially announced the incorporation of mathematical modelling in the Colombian mathematics education (Villa-Ochoa, Bustamante, Berrio, Osorio & Ocampo, 2009). In the official proposal issued by the Colombian government, it is said that mathematics education should contribute to the formation of citizens with the necessary skills for exercising their democratic rights and obligations. Within the proposal, mathematical modelling is considered as one of the five basic processes of mathematical activity to be studied in Colombian mathematics classrooms (Ministerio de Educación Nacional, 2006). Another example of this trend is Mexico. In 2006 a national reform of lower secondary education was initiated. This is the first reform for this educational level in which mathematical applications and its relation with other areas of knowledge are considered as a compulsory element in the curriculum. Thus, because of the existence of this institutional trend in Latin America advocating the inclusion of mathematical modelling in mathematical instruction, I thought it was relevant and important to discuss with the teachers some aspects of the teaching and learning of mathematical modelling.

The *didactical aim* of the course was to introduce teachers to mathematical modelling as a component of the teaching of mathematics. Particularly it was planned to discuss and illustrate through some activities, some of the arguments that have been provided to introduce this mathematical topic in the curriculum.

In order to structure the discussion about the arguments for teaching mathematical modelling, I relied on the paper Blum & Niss (1991). I was particularly interested in illustrating three of the of the five arguments presented in this paper, namely, the *promoting mathematics learning argument*, the *utility argument*, and the *critical competence argument*. I decided to illustrate only three arguments because I did not want to produce a course where teachers were overloaded with tasks. I selected the arguments that seemed easier to illustrate.

The *promoting mathematics learning argument* asserts that mathematical modelling can be a means of helping students to improve or reinforce their understanding of mathematical concepts and methods. The *utility argument* states that students should have the opportunity to explicitly study and practice mathematical modelling during their mathematical instruction. This argument is based on the assumption that the ability to apply mathematics in extra-mathematical contexts does not arise spontaneously. That is, in order to apply mathematics one should possess certain mathematical knowledge, but it is also necessary to practice how to apply such knowledge. The *critical competence argument* refers to the use of mathematical modelling as a means of educating citizens to identify, analyze and evaluate how mathematics is applied in society for decision-making or to find solutions to socially relevant problems.

The modelling course was structured around these three arguments. I think all of them are relevant to mathematics teachers regardless of the educational level in which they work. This was an important consideration since the teachers who participated in this course work at different educational levels: secondary level but also university level. All three arguments were illustrated through three activities. The course also included the reading and discussion of a research paper as an activity. The following table shows an overview of how the course was organised around these activities:

WEEK NUMB ER	ACTIVITY DEVELOPED	COMMENTS
1	Activity 1	The activity was individually solved and then discussed collectively in an asynchronous discussion forum
2	Activity 2	The activity was collectively solve in an asynchronous discussion forum
3	Activity 3	The activity was discussed together with the teacher educators
4	Activity 4	This activity was individually solved and then discussed collectively with the teacher educators in a asynchronous discussion forum
5	Closing stage of the course	This stage consisted of reading a research paper and watching a final video message

Table 2. Course structure overview.

In the next section the activities used to illustrate the three above-mentioned arguments are discussed.

4.2 The specific activities of the course

Before describing the content of the particular activities used in the course, it is important to clarify something. The two courses that were designed as part of the research method were aimed at fulfilling a double aim: their didactical aim and their scientific aim. In the case of the first course, which is discussed in this chapter, its didactical aim was to introduce teachers to mathematical modelling as a component of the teaching of mathematics. The introduction was based on the illustration of three of the existing arguments to include mathematical modelling as a component of mathematics instruction. On the other hand, the course was also designed to encourage interactions and promote the emergence of reflections in the mathematics teachers participating in the course. This was the *scientific aim* of the course. It was a difficult task to try to design activities that could help to simultaneously fulfil both aims. This was a tension that was always present in the design of the courses. It is possible that some activities have worked better to fulfil its didactic purpose than its scientific aim or vice versa. The reader should keep in mind these constraints when analysing the proposed activities.

4.2.1 Activity 1: Graphs representing movements

The first activity presented to the teachers was called "graphs representing movements". The *didactical aim* of this activity was to illustrate the *promoting mathematics learning argument*.

In the first part of the activity, teachers were asked to observe a four minutes long video. This video, called *V1*, shows a person (myself) using a motion probe and a graphical calculator as means of producing two graphical representations. These two graphs represent the two trajectories that I described while I was walking towards a wall and away from it. The purpose was to show to the teachers the relationship between the shape of the graphs produced by the technological devices, and the two trajectories of my movements. I expected the teachers to "read" in the graphs the characteristics of the represented trajectories. I interpret the activity that I develop in the video as a modelling activity, or at least as part of a modelling process. I interpreted it as part of a modelling process since I am using mathematical objects (the graphs) to represent a non-mathematical phenomenon (my movements). It is recommended to the reader to watch the above-mentioned video in order to obtain a clearer idea of the activity. Although the video is in Spanish, the reader can see the performed movements and the graphs associated with such movements. The video is stored on YouTube and can be accessed through the link: http://bit.ly/5avDUj

During the second part of the activity a didactical device called "note of reflection" was used (see Sánchez, 2008). A *note of reflection* is a written case study. Case studies (written cases, video cases and multimedia cases) are frequently used to assist teachers in examining their practice and their students' reasoning and understanding (Sowder, 2007, p. 180). What is particular about the notes of reflection is that they are written cases in which an *fictional situation* is described. The note of reflection is a useful tool that allows the teacher educator to focus teachers' attention and guide discussions towards particular issues that are considered relevant to address. The note of reflection helps teachers to analyse their practice, but also helps them to analyse other relevant aspects of their profession. Their mathematical knowledge is an example of such relevant aspects.

In the first activity the note of reflection was used to present a situation in which a mathematics teacher shows the video *V1* to her students. After showing the video, the teacher distributed among her students a document containing six graphs (see figure 5). Then the teacher tells her students that the six graphs possibly represent movements as those shown in the video *V1*. Then the teacher asks her students: how should you move in order to produce each of these six graphs, using the same technological devices that where used in the video?

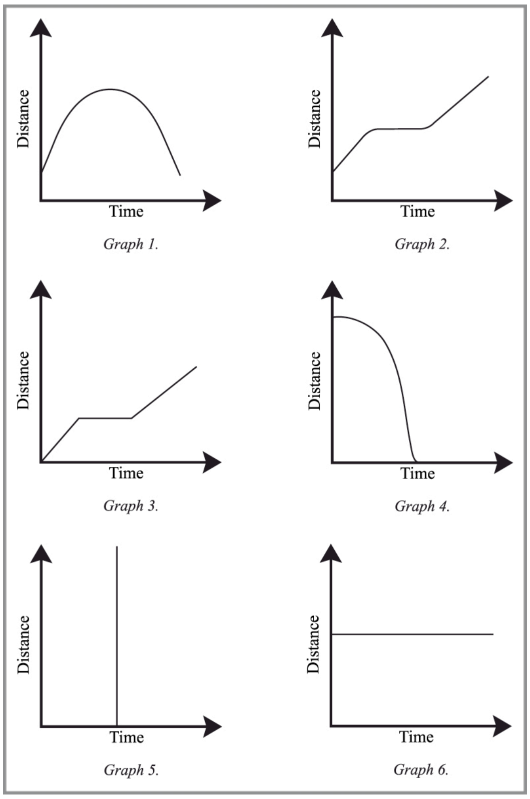

Figure 5. These are the six graphs included in the note of reflection. The Y-axis represents the distance from the wall. The wall is located on y=0.

The note of reflection also included the answers that two imaginary students, called Chuy and Mauricio, provided to the teacher in relation to her question. These are the answers provided by Chuy and Mauricio:

Chuy:

– In graph 1, a person who is away from a wall begins to move even further away from it. Then he reduces his velocity until he suddenly stop for a moment, and then he starts walking towards the wall.

– In the second graph, a person who is away from the wall starts to walk away from it. After that he stops for a moment and then he starts walking again but with a different velocity than the initial one.

– Graph 3 shows a person who is close to a wall and makes a movement in three stages: firstly he moves away from the wall, then he walks parallel to it, and finally he continues moving away from the wall.

– Graph number 4 shows a person who is away from the wall and walks parallel to it for an instant. Then he walks quickly towards it to finally touch it.

– In graph 5 the person should move away from the wall with a constant time.

– In the sixth graph the motion should be horizontal in order to have the same distance to the wall but keeping the time running.

Mauricio:

– In graph 1, you should move away from the wall and then come back.

– The second graph is very similar to the one from the video. You just need to move away from the wall, to stop, and then to keep walking.

– In graph 3, a person who is in front of a wall stars to walk towards it with a variable distance and time. Afterwards the person covers a distance with variable time but with a constant distance. Then he walks back to the wall with variable distance and time.

– In the fourth graph you should walk towards the wall, but the time should be squared every time you move.

– In the fifth graph a person should change his position with infinite velocity.

– In the last graph, a person walks towards the wall and just before reaching it, the person turns to the right and walks with variable time and constant distance.

The note of reflection concludes by asking the following questions to the mathematics teachers: Are Chuy and Mauricio's answers correct? Why?

Asking the teachers to evaluate the answers of the imaginary students is an indirect way to learn about teachers' interpretations of the graphs.

The correct interpretation of the graphs included in the note of reflection requires that teachers consider not only the path to be followed by the individual, but also his velocity. Graphs 2 and 3 for example have a very similar path, but the speed changes are more sudden in the graph number 3. Graphs 5 and 6 are the trickiest. In fact, it was expected that the interpretation of such graphs would cause difficulties for some of the teachers. In a study involving secondary and university level mathematics teachers, Dolores, Alarcón & Albarrán (2002) reported that teachers find difficult to interpret Cartesian graphs representing movements, particularly graphs as the number 5 and 6 included in figure 4. The expectation regarding teachers' interpretation of the graphs was also based on my own experience as a teacher educator. I had previously used this activity a couple of times with other groups of teachers. It is common to find teachers experiencing problems to assign physical meaning to some of the graphs, even in cases where the graphs were physically meaningful. Actually, the answers provided by the imaginary students Chuy and Mauricio and included in the note of reflection, are answers given by real mathematics teachers working with this type of tasks.

As I have already mentioned, through this first activity I tried to illustrate the promoting mathematics learning argument. I was trying to show to the teachers that technological-aided modelling of motion could serve as a means for introducing students to some mathematical concepts. For instance, in this context in which the shape of the graphs is defined by the nature of the movement that they represent, it would be easier to understand that the slope of a graph is related to the velocity with which the person moves. The activity could serve as a sort of intuitive introduction to mathematical concepts such as slope and derivative. My assumption that the modelling of motion could serve as a means for introducing students to mathematical concepts is based on the results obtained by different research. One example is the work of Ferdinando Arzarello and colleagues about the approaching of mathematical functions through motion experiments (see Arzarello & Robutti, 2004; Arzarello, Pezzi & Robutti, 2007). In these study researchers report the results of some teaching experiments aimed at the construction of the concept of function as a tool for modelling motion. Researchers claim that through this sort of approach students can develop competencies in describing mathematically a function. According to the researchers the students develop their knowledge through a process which starts from their perceptions and experiences with the motion sensor and evolves through interactions supported by gestures and natural language. Additional research results suggesting that the use of motion sensors can serve as a means for supporting students to encounter ideas such as distance, speed, time, and acceleration can be found in Nemirovsky, Tierney & Wright (1998).

This first activity was also aimed at promoting interactions and supporting the emergence of teachers' reflections. In particular, it was expected to foster the emergence of *mathematical reflections* on those teachers who experienced difficulties in interpreting the graphs included in the note of reflection. Teachers were expected to discover something new or to make visible some implicit aspects of their mathematical knowledge related to the interpretation of this sort of graphs. For instance, if a teacher misinterpreted any of the graphs, it was expected that he or she could reconsider his or her interpretations of the graphs, and hopefully realise that some of them may be incorrect.

To try to promote such mathematical reflections, the following steps were taken with respect to the organisation of the activity. Firstly, teachers were asked to individually answer the questions raised at the end of the note of reflection. Then, they should send their individual answers by email to the teacher educator in charge of the course (myself). This process was a confidential way of identifying which teachers had difficulties in interpreting the graphs. After analysing the individual answers, working groups of three and four members were defined. In the working groups the teachers who had difficulties in interpreting the graphs were mixed with those who had not experienced problems. At a later stage these groups should collectively discuss the activity 1 in an asynchronous discussion forum. The purpose here was to constitute discussion groups where different opinions and interpretations about the graphs could emerge. The decision about gathering together teachers having different perspectives on how to interpret the graphs was based on the assumption that facing or discovering interpretations that are different to yours, can lead you to revisit your own interpretations, and possibly trigger some reflections about your understanding.

Another reason to organise working groups constituted by people having different points of view (hereinafter called *heterogeneous groups*) is that such kind of groups tends to favour interaction. Research studies such as McGraw et al. (2007) and de Vries, Lund & Baker (2002) have pointed out heterogeneity of views as a factor favouring online dialogue and interaction. For example in McGraw et al. (2007) pre-service mathematics teachers, in-service mathematics teachers, mathematicians, and mathematics teacher educators are gathered together to analyse multimedia cases during online and face-to-face discussions. The different points of view of these people favoured the creation of a space for dialogue and discussion.

Additional measures were taken in order to encourage the emergence of reflections. The measures were inspired by the information obtained through the literature review on the concept of reflection, which was presented in chapter 3. One measure was to ask the teachers to carry out their discussions in asynchronous forums. In the particular case of this first activity, teachers should discuss the questions posted in the note of reflection during six consecutive days. At the end of the discussion teachers should produce a collective answer to the questions. An answer they could agree on.

The role of the asynchronous discussion forums in this research is quite important. Through them teachers were provided with several conditions that have been identified as favourable to the emergence of reflections[23], namely, (1) the requirement of *communicate their ideas in a written form*, and (2) the *availability of time* for posting their comments and consider the opinions of their colleagues. The communication through the exchange of written messages, and the availability of time to make comments, are two of the main characteristics that define an asynchronous discussion forum. The need to post written messages and opinions that will be permanently recorded and publicly accessible encourages people to review and carefully consider the content of their comments prior to posting them. This feature of the asynchronous discussion forums has been considered a powerful way of reflecting in previous studies such as Ponte & Santos, 2005 and Viseu & Ponte, 2009. In another internet-based study involving pre-service mathematics teachers, McDuffie & Slavit (2003) found that this kind of forums encourage the production of reflective comments:

[23] This conditions were already discussed in chapter 3.

> “The asynchronous nature of the discussion allowed PSTs [prospective teachers] to craft and edit their remarks prior to submission in a thoughtful manner. As in the earlier examples, PSTs' comments were often more focused and reflective than if spoken extemporaneously, or if even written privately in a journal. Further, the variety of feedback from peers led to directions for further thought that were quite impossible to receive from a single instructor or in a classroom discussion. This combination of factors both initiated and extended the reflective process in the PSTs” McDuffie & Slavit (2003, p. 462)

On the other hand, the availability of time that a discussion forum offers, allows participants to choose the time of day (or night) that is most convenient for them to participate in an academic discussion and it also allows them to: “take more time to think about the ideas of others and to craft their own responses before posting the online discussion” (Borba & Gadanidis, 2008, p. 197). These two features made me consider the asynchronous discussion forums as a design element that could encourage the emergence of reflections in mathematics teachers.

4.2.2 Activity 2: The paper airplane problem

The second activity I presented to the teachers was called “the paper airplane problem”. The paper airplane problem is an activity taken from the article Lesh & Caylor (2007), but that was slightly modified to adjust it to the purposes of my design. The context of the problem is a paper airplane contest where several pilots test some of the characteristics of the aircraft. In the original version of the problem included in Lesh & Caylor (2007), the problem solver is asked to determine the accuracy of the paper airplanes. In the modified version of the activity teachers are asked to answer the following question: Which one is the best airplane? I will explain the reasons for this modification later.

Teachers were introduced to the activity in the following way:

"The context of the problem is a paper airplane contest. Some flight characteristics that were tested are: (a) how far the planes flew, and (b) how long the planes stayed in the air. But, it was difficult to judge some of these characteristics because the planes performance depended on which "pilots" tossed them. So, next year, the organizers of the paper airplane contest have decided that three "pilots" should fly each plane, and that the same three pilots should fly all of the planes.

Your Task: Tables 2 and 3 show a sample of data from last year's contest. Based on this sample of data, please write a letter to the contest organisers and judges explaining how they could use these data and data from future contests to measure and decide **which one is the best airplane**.

Table 3 shows the results from a trial in which three pilots flew four different paper airplanes. The "pilot" stood at a point (0, -80) on the floor, and their goal was to toss the planes so that they came as close as possible to the point (0, 0) which was marked with an X, and which was the "target" for the flights.

Note: When testing planes, each plane was tossed by three pilots; and, for each toss, measurements were recorded about: (a) which pilot tossed the plane, (b) where the plane landed, (c) how close the landing point was from he target, (d) how far the plane landed from the starting point, and (e) how many seconds the planes were in the air."

After this introduction, teachers were provided with the following two tables:

	Plane 1			Plane 2			Plane 3			Plane 4		
	Flight	X	Y	Flight	X	Y	Flight	X	Y	Flight	X	Y
Pilot 1	1	45	−78	1	−37	−30	1	−42	−4	1	15	66
	2	7	−78	2	−48	−9	2	45	5	2	32	−46
	3	55	−42	3	28	26	3	35	−7	3	5	32
	4	−14	−46	4	12	−35	4	−13	5	4	14	−21
	5	21	−29	5	−19	26	5	4	−11	5	27	−5
Pilot 2	1	−12	26	1	−11	13	1	−4	12	1	−13	22
	2	−40	−20	2	−14	−36	2	24	9	2	−3	−21
	3	−38	22	3	31	13	3	−9	34	3	−11	−51
	4	−61	15	4	−43	10	4	−39	29	4	−12	56
	5	−48	61	5	−14	50	5	40	21	5	5	−69
Pilot 3	1	42	71	1	−9	6	1	24	41	1	53	−62
	2	61	38	2	−27	−13	2	−3	52	2	52	54
	3	43	27	3	17	−17	3	−31	54	3	48	−26
	4	18	50	4	14	47	4	5	64	4	45	25
	5	15	6	5	−36	41	5	36	63	5	25	40

Table 3. This table shows the coordinates of the landing points for each of the flights. The table was taken from Lesh & Caylor (2007).

Flight		Plane 1			Plane 2			Plane 3			Plane 4		
		Distance from target (inches)	Length of throw (inches)	Air time (sec)	Distance from target (inches)	Length of throw (inches)	Air time (sec)	Distance from target (inches)	Length of throw (inches)	Air time (sec)	Distance from target (inches)	Length of throw (inches)	Air time (sec)
Pilot 1	1	90	45	0.66	47.6	62.2	0.52	42.2	86.8	0.59	67.7	146.8	2.22
	2	78.3	7.2	0.58	48.8	85.7	0.91	45.3	96.2	0.70	56	47.7	1.42
	3	69.2	66.9	0.76	38.2	109.6	1.05	35.7	81	0.55	32.4	112.1	1.91
	4	48.1	36.8	0.51	37	46.6	0.51	13.9	86	0.61	25.2	60.6	1.51
	5	35.8	55.2	0.65	32.2	107.7	0.99	11.7	69.1	0.54	27.5	79.7	1.58
Pilot 2	1	28.6	106.7	0.94	17	93.6	0.95	12.6	92.1	0.73	25.6	102.8	1.88
	2	44.7	72.1	0.81	38.6	46.2	0.54	25.6	92.2	0.70	21.2	59.1	1.40
	3	43.9	108.8	0.93	33.6	98	0.92	35.2	114.4	0.77	52.2	31	1.35
	4	62.8	112.9	0.9	44.1	99.7	0.99	48.6	115.8	0.80	57.3	136.5	2.35
	5	77.6	148.9	1.28	51.9	130.8	1.10	45.2	108.6	0.78	69.2	12.1	0.97
Pilot 3	1	82.5	156.7	1.34	10.8	86.5	0.88	47.5	123.4	0.80	81.6	56	1.36
	2	71.9	132.8	1.14	30	72.2	0.73	52.1	132	0.88	75	143.7	2.39
	3	50.8	115.3	1.01	24	65.3	0.55	62.3	137.5	0.92	54.6	72.2	1.46
	4	53.1	131.2	1.21	49	127.8	1.08	64.2	144.1	0.89	51.5	114.2	2.18
	5	16.2	87.3	0.85	54.6	126.2	1.10	72.6	147.5	0.94	47.2	122.6	2.09

Table 4. This table shows the distance, time and flight sequence data for each pilot and airplane. The table was taken from Lesh & Caylor (2007).

To solve this activity, teachers were organised in groups. Teachers were asked to collectively solve the activity in a six-days long asynchronous forum. At the end of this period of time each group should deliver their collective answer to the paper airplane problem by email. The working groups had three of four members. Here again it was intended to form "mathematically heterogeneous groups" using the information obtained from the individual answers to the first activity. Teachers who had different interpretations about the graphs presented during the first mathematical activity were grouped together. However, the distribution of the teachers in the groups was different to the one applied during the first activity. The distribution was changed in order to give teachers the opportunity to work and interact with different colleagues during the course.

Through "the paper airplane problem" I intended to illustrate the utility argument. Particularly, the activity was used as a means of making teachers aware of the need for explicitly practising mathematical modelling in mathematical instruction. I think the activity is appropriate to achieve this purpose since it is necessary to go through some of the stages of a modelling process in order to solve it. As a consequence, the task allows teachers to experience, at least to some extent, the complexity that is involved in a modelling process. I will try to clarify this point by using as a reference the model of a mathematical modelling process presented in Blomhøj & Jensen (2003) that is shown in figure 6:

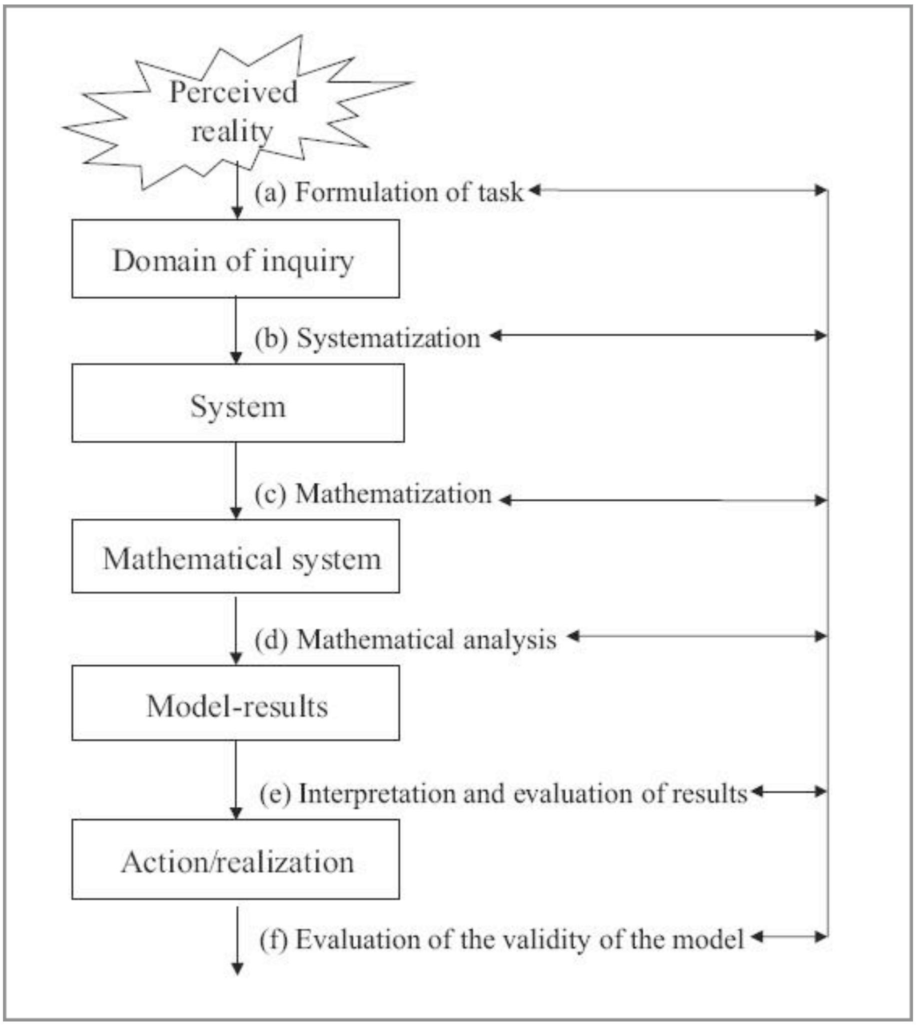

Figure 6. A graphic model of a mathematical modelling process. Diagram taken from Blomhøj & Jensen (2003).

In "the paper airplane problem" the stage (a) *formulation of task* of the modelling process is given. Teachers need to answer the question which one is the best airplane? but taking into consideration the data provided in the problem. Nevertheless, teachers need to transit through stage (b) *systematization* where they have to explicitly define or create a system that could allow them to determine which is the best airplane. Following this, teachers need to translate their system into a mathematical system. This step is interpreted as the stage (c) of the model. Teachers also need to carry out a *mathematical analysis* (d) where they should apply their mathematical system in the analysis of the data provided in the problem. The activity does not require that teachers pass through the stages (e) and (f) of the modelling process, although the formulation of the problem does not prevent them from doing so. The activity challenges the teachers to develop a prescriptive model. A prescriptive model which could be used to define the characteristics of a winning airplane for any contest.

The stage (b) *systematization* is a key stage in the activity. To answer the question “which one is the best airplane?”, teachers need first to define what does it means to be “the best”, and this definition should be connected with the provided data. At this stage, I particularly wanted teachers to note that in a modelling activity like this one, there are several possible and valid answers. The answer depends on the assumptions and considerations on which the model is based. To try to favour the appearance of different answers during the solution process of the problem, I decided to replace the original request of making judgements about the accuracy of the paper airplanes for the more general question: which one is the best airplane? I assumed that because of this question is more general and open, the teachers will propose different ways or criteria to identify the best airplane.

I assumed that by experiencing the undetermined nature of the initial parts of a modelling process (and in particular of the stage (b)), where “[o]ne is left with a feeling that can be characterized as 'perplexity due to too many roads to take and no compass given'” (Blomhøj & Jensen, 2003, p. 127), teachers will become aware and sensitive towards the need for explicitly practising mathematical modelling in mathematical instruction. I expected teachers to perceived that a modelling process is not a trivial task and therefore should be explicitly practised during mathematical instruction.

As in the activity 1, it was assumed that the multiplicity of opinions about how to define the best airplane would promote interaction among teachers, but the emergence of *mathematical reflections* within the interaction was also expected. In particular it was expected that teachers suggested more than one way of solving the task, and that this variety of possible solutions helped them to discover that the kind of answers that are obtained through a modelling process depend on the model applied and on the assumptions that underpin the model.

4.2.3 Activity 3: The marginalization index

"The marginalization index" is an activity that I designed for the purpose of illustrating the *critical competence argument*. Through this activity I tried to show to the teachers an example of how mathematical models can be used by a government to justify particular decisions that are socially relevant. More generally, I wanted to show them how mathematical models can form our perception of reality and even prescribe aspects of it.

In order to design this activity, I looked up examples of applications of mathematical models in decision-making within several agencies of the Mexican government. One of the most interesting examples that I found was the marginalization index, which is produced every five years by the National Council of Population of Mexico[24]. The marginalization index is a measure that the Mexican government uses to differentiate between states and municipalities according to the kind of shortages that their population is experiencing. Through this tool, the Mexican government tries to locate where the most disadvantaged and marginalized localities are. The

[24] www.conapo.gob.mx

government uses this information to decide where to focus their social development programs such as construction of schools, hospitals, wiring installation, water supplies, etc.

The marginalization index is based on a mathematical model that measures nine dimensions of the marginalization in a locality: percentage of illiterate population, percentage of population without complete primary education, percentage of population without toilet or drainage, percentage of population without electricity, percentage of population without access to piped water, percentage of private homes with a level of overcrowding, percentage of population living with a floor made of soil, percentage of population in localities with fewer than 5,000 inhabitants, percentage of the employed population with income less than or equal to twice the minimum wage.

After studying the mathematical model used to measure these nine dimensions, I concluded that the mathematics involved might be too complex for the teachers. I was afraid that, if teachers were asked to analyse the entire model, time would be consumed trying to understand the mathematics involved in the model, without having the opportunity of reflecting on the social consequences of its application. For this reason I decided to shorten to some extent the mathematical part of the activity. I tried to do so by giving the format of a "note of reflection" to the activity. The note of reflection allowed me to focus the analysis on specific aspects of the mathematical model, and at the same time to communicate to the teachers in an efficient way some of the problematic issues related to the model that I identified when I analysed it for the first time.

Thus, I presented the activity to the teachers as a note of reflection, which describes an invented teaching situation that takes place within a mathematics classroom. The note of reflection shows a dialogue between a teacher and her students, where the students express through their opinions some of my own criticisms towards the model. The content of the note of reflection is the following:

> *Susana is a mathematics teacher who has been discussing with her students the use of mathematical models in connection with social issues. Susana comes to her class with a new activity for her pupils:*
>
> **Susana:** Hello, I have decided that we will devote this lesson and the next one to analyze particular cases of the use of mathematics in the solution of social problems. I have found a good example of this kind of applications in the webpage of the National Council of Population of Mexico. I refer to the *marginalization index 2000*. Here I have a copy of it, but you can access it online through this link (*Susana writes in the blackboard the following web address*)
>
> http://bit.ly/AWNkq
>
> **Susana:** The marginalization index is a measure that is used to define and guide social policies. Let me read to you the introduction of the model[25]:
>
> "In accordance with its attributions and responsibilities, the National Council of Population (CONAPO) conducts studies and constructs indicators to know the socio-demographic, economic, social and cultural characteristics of the marginalized and vulnerable sectors of society, with the purpose of providing criteria and demographic considerations to the programs aimed at enhancing the capabilities and opportunities for people. In that sense, the estimation of the state and municipal marginalization indexes for the year 2000, which we bring to light in this publication, is an institutional contribution to the process of the demographic planning and the social and economic development of the country. The increasingly widespread

[25] The following quotation is a translation of an excert taken from the original document where the marginalization index 2000 is presented.

use of the marginalization index in both the planning processes, as in the allocation of budgetary resources of federal and state governments, has helped to strengthen the coordination of public policies oriented towards the improvement of the living conditions of the most disadvantaged populations, and to strengthen the distributive justice at the local level. We hope that the dissemination of the marginalization indexes, and the analysis presented, help to promote policies and programs aimed at strengthening distributive justice and reduce the profound gaps in regional development of the country, while stimulating reflection and developmental research on the matter"

Susana: To reflect, that is exactly what I want you to do here. We will form nine groups and each one will analyse one of the models that are included in the appendix C of the document, and that are utilised to calculate the nine socio-economic dimensions that constitute the marginalization index.

The aim of the activity is to try to understand what are the variables and concepts involved in the model, what are the assumptions underpinning the model and to reflect on the advantages and disadvantages of each model as a tool for representing the socio-economic phenomenon that is intended to capture. During the next lesson each group will explain their model and the advantages and drawbacks you have found.

Although the analysis will be focused on the appendix C of the document, you are free of consulting the whole document or any other source of information that you consider relevant for your analysis. Now we will form the groups and distribute the models…

Susana defines the groups, distributes the nine models among the groups, and the students begin to work on the task until the end of lesson. Two days later, Susana returned to her class:

Susana: Good morning. Are you ready to continue the activity of the marginalization index? Which team wants to begin presenting their findings?

Emma: We would like to start.

Susana: Good. Tell us about your model and the things you have found.

Carlos: We worked on the ninth socio-economic indicator, which measures the percentage of the employed population with income up to twice the minimum wage[26]. This indicator is obtained by using the following mathematical model (*Carlos writes the following expression on the blackboard*)

$$I_{i9} = \frac{P_i^{sm\leq 2}}{P_i^O} \times 100$$

Where:

$P_i^{sm\leq 2}$ is the part of the employed population, who receives less than two minimum wages.

P_i^O represents the total of the employed population.

Carlos: But we think the model is sort of weird...

Susana: Why?

Carlos: Well, the first thing we did in order to understand the model was to look for the definition of *employed population* which is located on page 174 in appendix C. We were very surprised by the fact that the definition considers as employed those persons aged 12 or older who have worked at least one hour, one week before the interview was conducted... even when they have not received payment for their work!

[26] In 2008 the minimum wage in Mexico was approximately 3.16 € per day.

Susana: That part of the definition may sound strange, but that is an assumption of the model.

Carlos: But this assumption has consequences. For example, the model could yield a small number which means that in the locality where the model was applied only few people earn twice the minimum wage or less, but...

Emma: But the number does not say anything about the children below the age of twelve who are working. Said otherwise, the model is not sensible to child labour and exploitation. In the digital library of the INEGI[27] we found a study called "Child labour in Mexico 1995-2002"[28]. The study estimates that 1.1 million of boys and girls between 6 and 11 years old are working in Mexico, often without receiving any remuneration for their work. Are those children not marginalized?

Susana: The information you have found and your comments are very interesting. Before discussing it further I would like to know if you have found anything else about the model.

Sandra: Yes we did it. We think it is also possible to have the opposite situation using the same model.

Susana: What do you mean?

Sandra: In the footnote located on the page 23 of appendix C, it is said that many people, especially those with the highest income, tend to omit information about their income. When I read this, I immediately thought on all the kidnappers, drug dealers and people working in the black market who have illegal income and are evading taxes... I am sure all those persons lie about their income.

Susana: And?

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[27] National Institute of Statistics, Geography and Computing of Mexico www.inegi.gob.mx

[28] Retrieved on march, 2008 from http://bit.ly/7ziSPw

Sandra: Then the model could generate a big number indicating that there are many people on a low income in this community, when they are actually rich people. For example, the municipality of Badiraguato in the state of Sinaloa is one of the largest producers of marijuana and poppy at the national level. Farmers are not going to declare what their real work is, or how much do they earn from doing that right? What is interesting here is that in the appendix B of the marginalization index it is claimed that in Badiraguato over 72% of the population earns less than twice the minimum wage and it is ranked as the most marginalized municipality in Sinaloa. I wonder if the model adequately represents the reality and if the political decisions based on this model are socially just.

Susana: Your analysis is really interesting, congratulations. Does anyone want to comment on the analysis made by Carlos, Emma and Sandra? Has anyone found something related to the rest of the models?

Your Task:

1. What is your opinion about the analysis of the ninth socio-economic indicator made by Emma, Carlos and Sandra?
2. Select one of the models used to calculate the nine socio-economic indicators and analyse it paying especial attention to its strengths and weaknesses (if any) as a tool for representing the social reality for which it is designed. Use mathematical arguments (numerical calculations, algebraic calculations, graphs, analysis of the effect of parameters and variables on the results produced by the model, etc.) to justify your analysis.

As in the activity 1, teachers were asked to individually answer the questions posted at the end of the note of reflection and send their answers to me by email. With question (1) I tried to bring the teachers to express some extra-mathematical reflections regarding the uses of mathematical models in society. That is, it was expected that the teachers expressed some type of surprise or revelation about the way in which mathematical models can be applied to support governmental decisions, and the type consequences that such application may produce. The aim of question (2) was to make teachers to extend the analysis presented in the note of reflection. It was expected that teachers could contribute to the analysis and discussion of the marginalization index by providing additional insights on the functioning of the mathematical models underpinning it.

There was a second stage of the activity where teachers were organised in groups to discuss the contents of the note reflection within an asynchronous discussion forum during five days. This forum was also attended by teacher educators. The role of this forum was to encourage the exchange of opinions among teachers about the contents of the third activity, and hopefully to serve as a space for the manifestation of extra-mathematical reflections.

I think the "marginalization index" activity is well suited to illustrate the *critical competence argument*. I think so because it provides an authentic example of the use of mathematical models in political decision-making. Furthermore the "note of reflection format" of the activity allows us to focus the discussion and analysis only on some aspects of the model, in order to facilitate the identification of some "side effects" produced by the application of the model (such as the insensitivity to child exploitation). The example clearly illustrates some of the repercussions that, according to Skovsmose (1990), are produced by the introduction of a mathematical model in the discussion of a non-mathematical problem (in this case the socio-political problem of how to allocate the resources destined to support the social development of a particular country):

1. The original problem is reformulated into a different kind of discourse (a mathematical discourse).
2. The group of people who could participate in the discussion of the problem and its solution becomes smaller and having a very specific composition (only those citizens with some mathematical knowledge would be able to discern and criticize the weaknesses of the model).
3. The debate changes its character, originating the inclusion of quantitative affirmations and arguments. As a consequence the debate tends to be dehumanised (Skovsmose, 2005), since the model can cause the illusion that we are treating with variables and quantities, and not with disadvantaged human beings.

4.2.4 The closing stage of the course

As I have explained throughout this chapter, the three activities included in this course were aimed at illustrating three arguments advocating the inclusion of mathematical modelling in mathematics instruction. However, during the design of the course I considered the possibility that the application of such activities might fail to illustrate such arguments. Because of this, it was decided to include in the final part of the course an institutionalisation stage, in which the three arguments and their relation to the proposed activities were explicitly addressed.

Two activities constituted the institutionalisation stage of the course. Firstly, during the last week of the course, teachers were asked to read the article by Blum & Niss (1991). Then teachers were organised in groups to discuss the contents of the paper in an asynchronous forum which lasted five days and that was moderated by the teacher educators who participated in the course. Secondly, an official closure of the course was video recorded. In the video, the designer of the course (myself), refers explicitly to the way in which the three proposed activities tried to illustrate the three arguments presented in Blum & Niss (1991). The video was lodged in *YouTube* and it was shared with the teachers the last day of the course.

In the next chapter some of the results obtained during the application of this modelling course will be presented.

5. Outcomes of the first online course

This chapter shows a characterisation of some of the online interactions that occurred in the modelling course described in Chapter 4. The interactions characterised are mainly those where instances of reflection were detected. Such characterisation is done by means of the application of the IC-Model of Alrø & Skovsmose (2002). It is argued that there are some communicative characteristics in the interactions that favour the emergence of reflections, such as evaluating and challenging acts. The chapter ends with a discussion on the potentiality and limitations that I identified when using the IC-Model as a tool for characterising online interactions.

In this chapter I will present and analyse the data I collected when applying the modelling course described in the previous chapter.

As already mentioned in the introduction, during the first stage of my research my attention was focused on the possible connections between the emergence of reflections and human interactions. I considered the *Inquiry Co-operation Model* (IC-Model) (Alrø & Skovsmose, 2002) as a viable theoretical tool to study these possible connections. This because the within the model interactions and reflections are considered as two connected processes. The model is based on the assumption that reflections arise from interpersonal interactions, which is a perspective consistent with my perception of the concept of reflection.

The IC-Model is a tool for characterising, from a communicative perspective, the type of interactions that occur when a group of people are faced with mathematical tasks. The idea was to use the model to characterise the interactions in which instances of reflections appeared, and then to locate the characteristics that were common for such interactions. These common characteristics may then be considered as factors favouring the emergence of reflections.

The chapter is divided into four parts. In the first part the IC-Model is briefly described. The second part refers to the general characteristics of the data analysed in this study. It is also described how these data are organised, selected and analysed. In the following part I will illustrate through three cases the way in which the IC-Model was used to analyse the data, and the type of results obtained. The chapter ends with a critical discussion on the potentiality and limitations of the application of the IC-Model within the specific context of my research.

5.1 Description of the Inquiry Co-operation Model

The Inquiry Co-operation Model was developed through empirical observations of face-to-face interactions between young students and their teachers, when the students were faced with open-ended mathematical investigations. One of the main assumptions underlying this model is that: "The qualities of communication in the classroom influence the qualities of learning mathematics" (Alrø & Skovsmose, 2002, p. 11).

According to these researchers, there are certain communicative characteristics that, when present in an interaction, define a special kind of interaction. This kind of interaction possesses the potential to serve as a basis for critical learning and reflection. This particular kind of interaction is called *dialogue* and its main qualities are that it should be based on mutual respect, on the willingness to make public our own ideas and subject them to scrutiny, and on a real interest to listen and analyse our interlocutor's ideas.

The communicative characteristics that define a dialogue are *getting in contact*, *locating*, *identifying*, *advocating*, *thinking aloud*, *reformulating*, *challenging* and *evaluating*. Such communicative characteristics can be succinctly defined as follows:

> "[*G*]*etting in contact* involves inquiring questions, paying attention, tag questions, mutual confirmation, support and humour. *Locating* has been specified with the clues of inquiring, wondering, widening and clarifying questions, zooming in, check-questions, examining possibilities and hypothetical questions. *Identifying* involves questions of explanation and justification and crystallising mathematical ideas. *Advocating* is crucial to the particular trying out of possible justifications, and it is closely related to arguing and considering. *Thinking aloud* often occurs as hypothetical questions and expression of thoughts and feelings. *Reformulating* can occur as paraphrasing, completing of utterances and staying in contact. *Challenging* can be made through hypothetical questions, examining new possibilities, clarifying perspectives, and it can be a turning point of investigation. *Evaluating* implies constructive feedback, support and critique" (Alrø & Skovsmose, 2002, p. 110).

The fact that within the IC-Model interactions and reflections were conceptualised as two related elements, it made me perceive the model as a viable tool to try to identify the possible relationships between interactions and reflections. However, I also had some doubts about the feasibility of implementing the IC-model in an online setting. I was particularly intrigued by the fact that in their data analysis, the authors make interpretations on the communicative characteristics of an interaction based on elements that are not necessarily perceptible in an online setting, such as gestures and tones of voice.

I did some tryouts in order to test the applicability of the IC-Model in an online setting. I chose one of the online in-service courses I designed and applied in Mexico before starting this PhD project, and used the categories of the IC-Model to analyse some of the interactions that took place in that course. Such these "applicability tests" were reported in

Sánchez (2008) and Sánchez (to appear, a). I discovered that it was indeed possible to distinguish within the data the communicative characteristics of the IC-Model, although some of them, as *locating* and *identifying*, were difficult to distinguish at times. There were utterances in the online interactions that, according to my interpretation of the model, they could be classified as locating acts, but also as identifying acts. In the section 5.5 of this chapter I present an example to illustrate this kind of difficulty. However, in order to help the reader to follow me in the data analysis, I must clarify that I considered as *locating acts* those utterances where some sort of discovery was expressed by the teachers, regardless of whether they expressed it in mathematical terms or not. The utterances where only mathematical ideas where expressed and clarified, but without expressing any sort of discovery, were classified as *identifying acts*.

5.2 On the nature and structure of the data

The empirical data that have been considered in this research are mainly online discussions held by mathematics teachers in asynchronous discussion forums. Such data have very specific characteristics that offer advantages but also impose restrictions on the data analysis.

One of the main features is that asynchronous discussions are carried out through the exchange of written messages, which can be complemented with images, links or other sort of attachments. When one of these messages is published, the message is permanently registered in the forum indicating the author, the date and time it was posted. The permanent record of these messages makes easy to store the data and facilitates the access to them at any time and place where an Internet connection is available. These features create a sort of "collective memory" (de Vries, Lund & Baker, 2002) with a high potential for producing reflections among the participants of a dialogue.

In addition, an asynchronous discussion offers a certain flexibility of participation that does not exist in a face-to-face discussion. The participants can contribute to the discussion at any time of day or night, plus they have time to analyse the comments of their colleagues and issue their own contributions. Another aspect of the flexibility of participation is the fact that an asynchronous discussion can be composed of several sub-discussions that participants can attend simultaneously. This situation makes no sense in a face-to-face setting since it requires to maintain two or more conversations, with different partners, but in a simultaneous way. This feature has been previously named "multi-logue":

> "By multi-logue, we mean the occurrence of various intersecting dialogues, as takes place in chat rooms, where members are involved in various discussions simultaneously, and a given individual 'skips' from one discussion to another" (Borba & Villarreal, 2005, p. 173, emphasis in the original).

A multi-logue is a manifestation of the flexibility of participation that the online settings offer to their users. However, this freedom that allows the participant to "jump" from one sub-discussion to another tends to generate untidy discussions that may be difficult to reconstruct and interpret for the outside observer (the researcher for example). I will try to illustrate this complexity by describing the general structure of an asynchronous discussion in the section 5.2.1.

Another relevant characteristic of asynchronous discussions is that the need for making transcriptions of the dialogues disappears, because they are originally produced in a written form. This not only means that the technical work of the researcher is lightened; it also means that the researcher gets a more naturalistic access to the observed phenomena. For instance, when one analyses face-to-face interactions by means of transcripts, what we are actually analysing is a representation of these interactions (the transcripts). The asynchronous discussions can be studied in a more direct way, reducing the possible discrepancy between the actual dialogue and the object of analysis.

5.2.1 Structure of an asynchronous discussion

I will try to give the reader an idea of the appearance and structure that an asynchronous discussion can take. I will use as an example one of the discussion forums in which teachers discussed the first activity of the modelling course. Upon entering the discussion forum of one of the working groups, one can see a table with four columns (see figure 7). The first column called “Tema” (theme or topic in English) indicates that the discussion was divided (by the teachers themselves) in three themes; the second column called “Comenzado por” (started by) indicates who started each of these themes; the third column named “Respuestas” (replies) shows the number of responses or comments included in each theme; and the fourth column called “Último mensaje” (last message) displays the date and author of the last published comment within a particular theme.

Tema	Comenzado por	Respuestas	Último mensaje
Primera aportación	Rebeca Flores Garcìa	36	Héctor Hernanez Guzmán mar, 1 de abr de 2008, 10:36
observación.	maria del rocio lammoglia lemini	24	maria del rocio lammoglia lemini lun, 31 de mar de 2008, 23:08
Análisis de las gráficas de movimiento	JOSE LUIS REY	19	maria del rocio lammoglia lemini lun, 31 de mar de 2008, 10:52

Figure 7. An asynchronous discussion can be composed of several sub-discussions. This figure shows three sub-discussions within an asynchronous discussion in which there have been respectively 36, 24 and 19 responses.

If I select the first theme named "Primera aportación" which contains 36 comments, I will join a new space where the top shows the comment that inaugurated this theme, followed by 36 replies or comments (see figure 8). All the comments indicate the date and time in which they were posted. As it can be noted, some of these comments are nested (see for example the highlighted section in figure 8). This indicates the existence of a sub-discussion, i.e. a string of related comments and responses. Thus, Figure 8 shows that within the "Primera aportación" theme there were at least five sub-discussions.

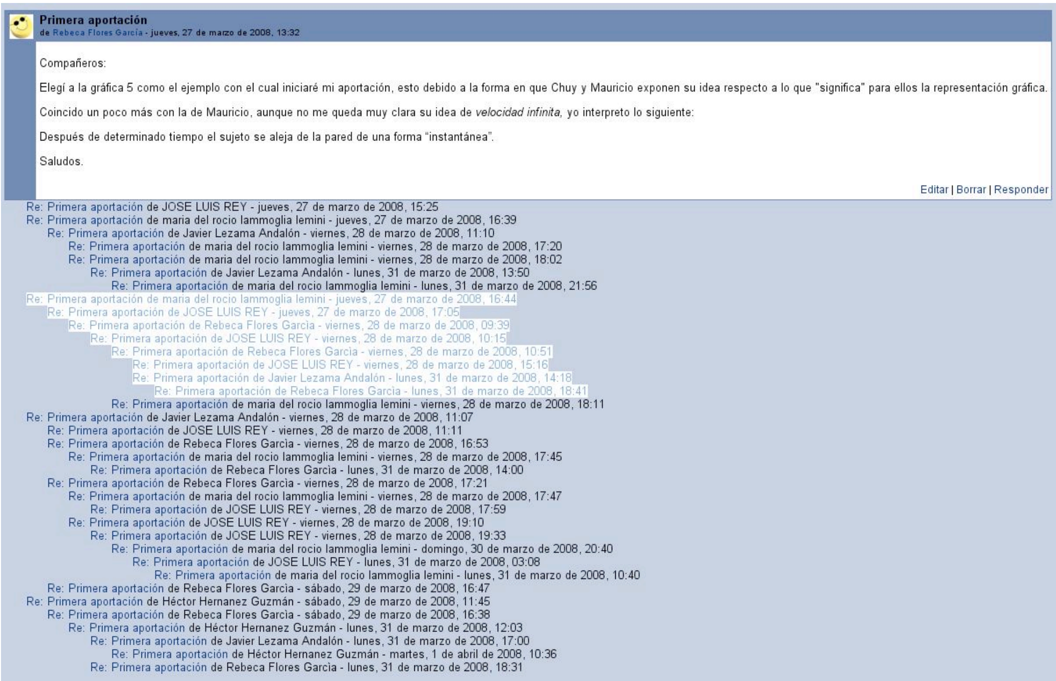

Figure 8. A sub-discussion is integrated by several individual utterances that are chronological tagged. Groups of utterances can be related or linked. The highlighted area indicates one of these groups of linked utterances.

As one begin to review the individual comments that constitute a sub-discussion, it is common to find attachments or links to complementary resources (such as text files and external webpages) that serve to illustrate or complement the ideas presented in the comments. One must then also review these additional resources in order to really understand the ideas expressed in the comments. What I want to convey to the reader is the idea that the analysis of online discussions can be a complicated task. It implies trying to grasp the meaning of discussions that are multilayered and ambulatory. However, it is an achievable task. Next, the method I followed to analyse this type of discussions in the context of my own research will be described.

5.2.2 Method for selecting and analysing the data

My method for selecting and analysing data can be conceptualised as consisting of three stages:

(1) The acquaintance stage. The overall purpose of this stage was to get a general idea of what happened in each discussion forum. To do this it was necessary to read and read again several times each utterance of a discussion (including its links and attachments), in order to become familiar with its contents. During this process I always kept in mind that the particular purpose of this stage was to locate instances of reflection. So, while I was trying to get familiar with the contents of a specific discussion, I also focused on locating the moments of an interaction that could be labelled as reflections, according to my own definition of the concept, as presented in chapter 3. Here the concept of reflection functions as a tool for the selection of data relevant to the research.

(2) The reconstruction stage. Once I have located the instances of reflection within the online discussions, in the second stage I focused on recreating the context in which these interactions occurred. That is, I tried to track down "the route" that a teacher followed and that led him/her to a reflection. I was trying to locate the people that the teacher interacted with during the prior moments to the appearance of the reflection. At this stage the chronological information that is included in each utterance was quite useful (see figure 8). By using this information it was possible to arrange in a chronological order the utterances issued by a specific teacher. In this way I could track her trajectory, regardless of whether the teacher was jumping from one sub-discussion to another.

(3) Application of the IC-Model. After recreating the interactive context in which the reflections emerged, the next step was to characterise such interactive context in terms of the IC-Model. That is, to identify the communicative characteristics that were present in the interaction in which the reflection emerged. This stage deserves a detailed discussion because it serves to exemplify the application of the theoretical concepts in the data analysis. Therefore in the next section I will illustrate, by showing some cases, how the IC-Model was applied.

5.3 Application of the Inquiry Co-operation Model

To apply the IC-Model is necessary to have located and delimited the interactive context in which a reflection has emerged. The application of the IC-Model consists then in going through all the interaction and associating each of its utterances with the communicative characteristics of the IC-model. Thus, all the utterances that I will present next have been labelled with the names of the communicative acts included in the IC-

Model. To facilitate their identification, those labels were written using italics.

All the displayed utterances have been translated from Spanish into English. The original names of the teachers involved in the interaction have been replaced in order to protect their identity. In some cases, I will use bracketed ellipsis [...] to denote the omission of certain segments of text. This edition was made for the sake of brevity and to increase the readability of the data. Each utterance has been numbered to facilitate its quick reference.

5.3.1 Case 1: A mathematical reflection

As a first example I will present a case that I interpret as containing an instance of a *mathematical reflection*, that is to say, it contains a moment in which certain aspects of the mathematical knowledge of a teacher are challenged and as a consequence, the teacher consciously reconsidered them. This instance of reflection appeared early in the development of the course, precisely when the teachers were confronted with the first mathematical activity of the modelling course, which was discussed in section 4.2.1 of the previous chapter.

In the previous chapter I mentioned that, before collectively discussing the first activity in the forum, teachers were asked to email me their evaluations of the answers provided by the imaginary students Chuy and Mauricio. At this stage I identified a teacher called Alberto who evaluated the student's answers in the following way (to complement the quotation see figure 9):

> "Regarding graph 5: Chuy is right when he says that the person is moving away [from the wall], but not when he refers to the constant time. On the other hand, Mauricio talks about an 'infinity velocity'. Maybe he means that the person is doing it very fast and this corresponds with the graph" (Quotation extracted from Alberto's individual report).

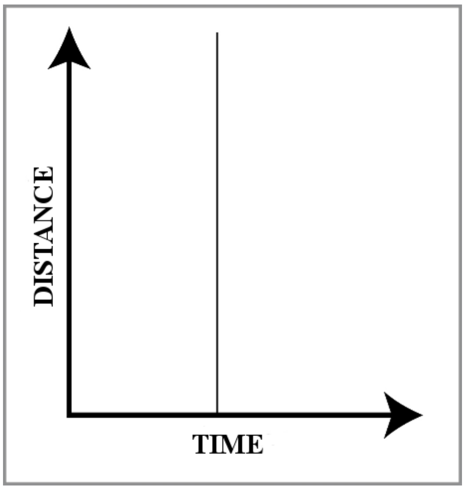

Figure 9. Graph number 5 included in the first activity.

When I read the previous quotation, I got the impression that Alberto thought that figure 5 may indeed represent some kind of physical movement. However, the graph number 5 is a graphical representation of a relationship that has no meaning neither in a physical context nor in a mathematical context. In a physical context it would be necessary to have a body occupying several different positions in a single instant of time. In a mathematical context one can argue that the graph 5 cannot represent a real function in one variable since there is an element in the domain of the function, which corresponds with more than one element in the codomain.

I decided then to place Alberto on a working group that included colleagues having a different interpretation of this graph. In total, this group consisted of four teachers. They discussed the first activity during six days in an asynchronous discussion forum. Alberto himself introduced the very first discussion theme of this forum. He opened the discussion with the following comment, which is quite similar to the one include in his individual report:

[1]
Theme: The first contribution
From: Alberto
Date: Thursday, 27th of March 2008, 10:05

[...]

Graph 5. Chuy is right by saying that the person is moving away, but not when he refers to the constant time. On the other hand Mauricio talks about an 'infinity velocity'. Maybe he means that the person is doing it very fast and this corresponds with the graph. Did he jump?

[...]

As mentioned before, apparently Alberto thinks that it is possible to produce the graph number 5 if a person moves very quickly or jumps, but in fact it is not possible to produce this graph using the motion sensor, neither is it mathematically coherent if we interpret it as a mathematical function. However, what is important to notice here is Alberto's attitude: to express our ideas and beliefs about a specific topic in an open way (as Alberto did) is considered as a *thinking aloud* communicative act.

Susana produced the first reaction towards Alberto's comment. She did not agree with Alberto's ideas:

[2]
Theme: Re: The first contribution
From: Susana
Date: Thursday, 27th of March 2008, 12:27

[...]

Graph 5. Here you will notice that I disagree with you Alberto because the explanation given by Chuy sets up an impossible situation, because it is not possible for a person to be in different places at the same instant of time t. I mean, to be away and close from the wall at the same time. It is not a mathematical function, and it does not make sense physically.

[...]

In [2] Susana is *getting in contact* with Alberto, that is to say, she makes explicit reference to Alberto's comments and she makes some remarks about it. In fact some of these remarks could be viewed as an *evaluative* act, because she explicitly explains why she does not agree with Alberto's interpretation of graph number five. After Susana's participation, Mariana joined the discussion:

[3]

Theme: Re: The first contribution
From: Mariana
Date: Thursday, 27th of March 2008, 15:47

[...]

I have read your comments regarding graph 5. In my opinion none of the students provided the right answer, this position is similar to your answer Susana. The idea of giving a big jump will not be represented by a vertical line; in this case it would have a negative slope, with an angle very close to a right angle but never perpendicular to the X-axis.

[...]

In her utterance [3] Mariana is also *getting in contact* with Alberto and Susana. In an *evaluative* act, she rejected the idea of the jump suggested by Alberto as a possible interpretation of the graph number 5.

It is important to note that so far, Susana and Mariana have kept the contact with Alberto by "listening" and analysing his comments. Both teachers have expressed, by means of evaluative acts, the reasons why they do not agree with Alberto's initial stance regarding the interpretation of the graph. I argue that this disposition to listen, to analyse and to evaluate Alberto's ideas is an indicator that these teachers have established a *dialogue* in terms of Alrø & Skovsmose (2002).

Another factor that makes me interpret this interaction as a dialogue is that Alberto did not feel inhibited by his colleagues' remarks; on the contrary, they served as an incentive to revisit and reconsider his own ideas. In a different sub-discussion, which Alberto started within the same discussion forum, he expressed:

> [4]
>
> *Theme:* Graph 5
> *From:* Alberto
> *Date:* Thursday, 27th of March 2008, 19:19
>
> Hello everybody
>
> Reflecting on graph 5, it does not make sense physically… and theoretically it would be impossible. We can see that the slope of the straight line is indefinite, because it reaches a value of 90°.
>
> Taking the slope formula as the velocity of this graph, distance versus time, we have that V=m=(d2 – d1)/(t2 – t1). Graphically we can see that there is a displacement, but the time doesn't change, it is the same. So: t2 – t1=0. Carrying out the division, we have that:
>
> V=m=(d2 – d1)/0, and this is indefinite.
>
> Therefore, I think there is no such behaviour that could produce a graph like this with the "motion sensor".
>
> What do you think colleagues?
>
> Alberto

I interpret [4] as the outcome of a *mathematical reflection*, where the mathematical reflection is the process through which Alberto changed his mind regarding the interpretation of graph 5. My claim here is that the remarks provided by Susana and Mariana, constituted a valuable reflection opportunity for Alberto. An opportunity to review his own comments and mathematical ideas and to try to verify their validity. It seems to me that in [4] Alberto *located* a mathematical structure that helped him to notice the impossibility of producing such a graph.

It is interesting to note that apparently Alberto was not the only one discovering something new by means of the interaction. Susana in [5] is *locating* a new way to justify the impossibility of producing the graph number 5:

> [4]
> *Theme:* Re: Graph 5
> *From:* Susana
> *Date:* Thursday, 27th of March 2008, 21:48
>
> Alberto: Of course I agree with your way of analysing the situation. I had not considered it from a theoretical point of view, taking into account the concept of average velocity and the variations of time and distance. I thought that using the concept of mathematical function would be enough, because it is not a function since for a value of t you have more than one ordinate value.
>
> What do you think colleagues? As always your comments and different point of views are welcome.
>
> [...]
>
> Susana

One hypothesis that arises after the analysis of this interaction is that a critical attitude, like the one Mariana and Susana expressed through their evaluative acts, may serve as a basis for the emergence of reflections. Next, two new cases that add evidence supporting this hypothesis will be presented.

5.3.2 Case 2: An interaction where no reflections appeared

The case I will present now (case 2) and the one I will present afterwards (case 3), are linked to the second activity of the modelling course called "the paper airplane problem". As I mentioned in chapter 4 (see section 4.2.2), I planned to let the teachers experience part of the complexity of a modelling process through this task. In particular I expected the mathematics teachers to propose different ways to solve the problem, and as a consequence to experience a state of puzzlement due to the different routes available to cope with the problem. Such situation only occurred in one of the working groups (out of four).

When I noticed this, I decided to compare the communicative characteristics of the interaction within this group with the interactions in those groups in which the activity did not work as expected. I found significant differences between both types of interactions. The cases 2 and 3 illustrate such differences. In the case 3 I will show the analysis of the interaction of a group where some reflections emerged; whereas in the case 2, I will show the analysis of the interaction of a group in which no instances of reflections were manifested. I will begin with the case 2.

This working group consisted of four mathematics teachers. One of them, Lucas, could not participate in the interaction due to personal reasons. Of the remaining three teachers, two of them were more actively involved in the discussion.

The interaction in this group started with a message posted by Juan in the forum:

[6]
Theme: The first message
From: Juan
Date: Tuesday, 1st of April 2008, 9:03

Sandra, Lucas and Horacio

First of all, hello to everybody, and let's get to work. I have just glanced through the activity 2 and we have much work ahead. I think it would be interesting if we could have read the literature for tonight and then try to communicate among us on the activity, and addressing our response as a group. Again we are against the clock, so don't give up .... !

Juan

Some hours after, Sandra posted her first comment in the forum:

[7]

Theme: Re: The first message
From: Sandra
Date: Tuesday, 1st of April 2008, 23:38

Hello Juan, Horacio and Lucas

You know, I have read several times the activity 2, and I got this questions: How do you determine who is the winner of this competition? Perhaps I would recommend to the organisers to reward the fastest plane, and the one staying longer in the air, and the one travelling the longest distance (although the goal is to reach the point (0,0)), and the best pilot, by establishing different awards in different categories.

Sandra

The next day, Horacio published his first message:

[8]

Theme: Re: The first message
From: Horacio
Date: Wednesday, 2nd of April 2008, 10:15

Hello Sandra, Lucas and Juan

I am on the runway

Best regards

Horacio

These three initial messages are very different. Sandra is the first one *thinking aloud* about the mathematical activity. In her message she expresses a doubt that arose after reading the activity. Her question is relevant because it addresses the *systematization*[29] stage of a modelling process, which is essential to determine the best plane of the competition.

However, none of Sandra's colleagues *made contact* with her regarding her question. That is, no one made reference to her question, nor provided ideas to discuss what could be meant by the best plane. Look for example at the second intervention of Juan in the forum:

> [9]
> *Theme:* Re: The first message
> *From:* Juan
> *Date:* Thursday, 3rd of April 2008, 9:57
>
> Sandra, Horacio and Lucas
>
> I started to think on the activity.
>
> I did some graphs using Excel. First I tried with all the results in a x-y diagram but I got 4 clouds of points and they do not help me to visualise anything. Then I tried with one graph for each plane.
>
> Afterwards I calculated the arithmetic mean of the distances to the origin for each flight. This would allow me to establish which plane was closer to the target on average (I found that it is the plane 2)
>
> But I feel it loose and unconnected. I need to find something more solid to prove which one is the best.
>
> I have not worked yet with the other options (I mean, flight time, distance). Please express your opinion to start to organise this [...]
>
> Juan

Juan *does not get in contact* with Sandra in [9]. Instead, Juan makes public the way he has begun to address the activity and his own perception of this approach. Thus, this utterance is classified as a *thinking aloud* act.

Sandra reacted to Juan's utterance in [10]. Basically she *gets in contact* with him by supporting his idea of considering other variables (flight

[29] See figure 6, chapter 4.

time, distance) but without taking into account the pilot's performance in the analysis. She does not mention any more her question about how to determine the winner:

[10]

Theme: Re: The first message
From: Sandra
Date: Thursday, 3rd of April 2008, 11:10

Hello Juan, Horacio and Lucas

Before sending something I was also trying and it seems not feasible to use a x-y graph. Your idea about the graphs for each plane is more logical, because the focus of the contest is the planes and not the pilots.

Now we have to find a way to use the three variables presented in table 2 and establish who the winner is. I am sure there are several ways of proving it. [...]

Sandra

[11]

Theme: Re: The first message
From: Juan
Date: Thursday, 3rd of April 2008, 11:40

Colleagues

One possible option is to work with some sort of weighted mean for the 3 considered variables (distance flown, distance from target and flight time). I think the most important is the closeness to the target. Another option is to consider the deviation of each landing point from the target (because it is definitely a measure of [statistical] dispersion)

What do you think?

[12]

Theme: Re: The first message
From: Juan
Date: Thursday, 3rd of April 2008, 11:51

Colleagues

An extra thing

I just used Excel to calculate the standard deviation for each of the planes and I found that the one with the smaller deviation is the plane number 2. The results were

43.4 28.1 28.6 37

Even though the difference between [planes] 2 and 3 is not too big, the result is plane 2 whether the arithmetic mean or the deviation is used (taking as a main variable the distance from target)

Now I will work on the other two variables

In [11] and [12] Juan is *identifying*, I mean, he is trying to clarify or crystallise his mathematical ideas. He is making specific suggestions on how to relate the three selected variables (flight time, length of throw and distance from target). He proposes to use a weighted mean where "distance from target" should be the most important variable. He also suggests using the standard deviation as an alternative way of measuring the proximity to the target. Here it is important to notice that Juan's calculations are wrong. The numerical values presented in [12] do not correspond with the standard deviations of the planes regarding their distances from target. The correct values would be 20.65, 12.44, 18.17 and 18.77. I am highlighting this because none of Juan's teammates noticed this kind of details, which resulted to be one characteristic of the interaction within this group.

[13]

Theme: Re: The first message
From: Sandra
Date: Thursday, 3rd of April 2008, 13:05

[...]

I was going to ask if you had thought of a linear regression. But I read your suggestion about the weighted mean, now we just need to decide which variable will be more important than the rest. Because the target is the point (0,0) the distance from target would get 40%, while the rest would get 30% each, if you agree.

After calculating the deviations, it seems that plane 2 is winning...Will this one be the winner according to our interpretation of the situation?

[…]

Sandra

[14]

Theme: Re: The first message
From: Juan
Date: Thursday, 3rd of April 2008, 19:06

Colleagues

I have been doing a sketch of the things we have done so far and I captured it on this draft I am attaching

Let me know what do you think (if it is too bad please be benevolent), I accept criticism but kind

Please have a look at it in order to provide suggestions, modifications, additions, etc., because the time is running out and we are about to land...

Juan

In [13] Sandra *keeps the contact* with Juan by referring to his proposal of the weighted mean. In her utterance Sandra mentions the possibility of using a linear regression, but this option was not further explored because she just dropped out this alternative to follow Juan's proposal about the use of a weighted mean. Without a completely clear argumentation, Sandra proposed specific weights for each element of the weighted mean.

In turn, Juan in [14] contributes to *not locate* Sandra's idea of a linear regression. In his utterance he completely ignores the timid suggestion of Sandra and he only "heard" the proposal of the specific weights. In a text file attached to his utterance number [14], Juan *identifies* or clarifies his perspective on the weighted mean. In this file he defines the concept of "performance" that could be use to determine which plane is the best one. The plane that gets the higher performance will be the winner. This concept is defined as follows:

$$Performance = 0.4x + 0.3y + 0.3z$$

Where

x = The arithmetic mean of the distances from target

y = The arithmetic mean of the lengths of the flights

z = The arithmetic mean of the flight times

The "performance model" was the one this group used to determine the best plane of the contest. When establishing this model, Juan never questioned the reasons underlying the weights suggested by Sandra, that is, he did not ask what were the assumptions that Sandra considered in order to establish these values. He just included these values in the model.

In general, the interaction between Sandra and Juan could be described as uncritical. They experienced a short and "smooth" interaction, where they did not question nor evaluate the proposals from the other. It was an interaction characterised by the absence of *evaluative* acts.

Thus, even though this working group was able to successfully solve the mathematical modelling task, i.e. to establish a model to select the best plane, the interaction within the working group was characterised by a poor exchange of perspectives and ideas on how to address the mathematical task.

5.3.3 Case 3: Mathematical and extra-mathematical reflections

This working group consisted of three mathematics teachers; nevertheless one of them practically did not contributed to the discussion. This teacher declared to have personal problems that prevent her from participating in a more active way in the forum.

The two remaining teachers, were very active during the discussion of the activity in the asynchronous forum. After the exchange of some greetings messages, one of the teachers called Nadia started to *think aloud* about the problem. In a file attached to one of her comments in the forum, Nadia *identified* a way of coping with the problem. She started by adding up the results achieved by each pilot in each category (flight time, length of throw and distance from target). Thus, for instance when referring to the category "distance from target" for the plane 1, Nadia calculated:

$90 + 78.3 + 69.2 + 48.1 + 35.8 = 321.4$ for pilot 1

$28.6 + 44.7 + 43.9 + 62.8 + 77.6 = 257.6$ for pilot 2

$82.5 + 71.9 + 50.8 + 53.1 + 16.2 = 274.5$ for pilot 3

Of these three quantities (321.4, 257.6 and 274.5), she selected the smallest one as representing the best score for this plane (in this case 257.6). I suppose she selected the smallest quantity because one of the aims of the competition was to land as close as possible to the target. In the case of the variables "flight time" and "length of throw", she also performed the individual sums of the marks obtained by each pilot, but in both cases she selected the biggest quantities as representing the best scores of the plane in these categories. Nadia condensed all this information in a table shown in Figure 10.

Pilot	Flight	PLANE																			
		1					2					3					4				
		x	y	Distance from target	Lenght of throw	Flight time	x	y	Distance from target	Lenght of throw	Flight time	x	y	Distance from target	Lenght of throw	Flight time	x	y	Distance from target	Lenght of throw	Flight time
1	1	45	-78	90	45	0,56	-37	-30	47,6	62,2	0,52	-42	-4	42,2	86,8	0,59	15	66	67,7	146,8	2,22
	2	7	-78	78,3	7,2	0,58	-48	-9	48,8	85,7	0,91	45	5	45,3	96,2	0,7	32	-46	56	47,7	1,42
	3	55	-42	69,2	66,9	0,76	28	26	38,2	109,6	1,05	35	-7	35,7	81	0,55	5	32	32,4	112,1	1,91
	4	-14	-46	48,1	36,8	0,51	12	-35	37	46,6	0,51	-13	5	13,9	86	0,61	14	-21	25,2	60,6	1,51
	5	21	-29	35,8	55,2	0,65	-19	26	32,2	107,7	0,99	4	-11	11,7	69,1	0,54	27	-5	27,5	79,7	1,58
2	1	-12	26	28,6	106,7	0,94	-11	13	17	93,6	0,95	-4	12	12,6	92,1	0,73	-13	22	25,6	102,8	1,88
	2	-40	-20	44,7	72,1	0,81	-14	-36	38,6	46,2	0,54	24	9	25,6	92,2	0,7	-3	-21	21,2	59,1	1,4
	3	-38	22	43,9	108,8	0,93	31	13	33,6	98	0,92	-9	34	35,2	114,4	0,77	-11	-51	52,2	31	1,35
	4	-61	15	62,8	112,9	0,9	-43	10	44,1	99,7	0,99	-39	29	48,6	115,8	0,8	-12	56	57,3	136,5	2,35
	5	-48	61	77,6	148,9	1,28	-14	50	51,9	130,8	1,1	40	21	45,2	108,6	0,78	5	-69	69,2	12,1	0,97
3	1	42	71	82,5	156,7	1,34	-9	6	10,8	86,5	0,88	24	41	47,5	123,4	0,8	53	-62	81,6	56	1,36
	2	61	38	71,9	132,8	1,14	-27	-13	30	72,2	0,73	-3	52	52,1	132	0,88	52	54	75	143,7	2,39
	3	43	27	50,8	115,3	1,01	17	-17	24	65,3	0,55	-31	54	62,3	137,5	0,92	48	-26	54,6	72,2	1,46
	4	18	50	53,1	131,2	1,21	14	47	49	127,8	1,08	5	64	64,2	144,1	0,89	45	25	51,5	114,2	2,18
	5	15	6	16,2	87,3	0,85	-36	41	54,6	126,2	1,1	36	63	72,6	147,5	0,94	25	40	47,2	122,6	2,09
Piloto 1				321,4	211,1	3,06			203,8	411,8	3,98			148,8	419,1	2,99			208,8	446,9	8,64
Piloto 2				257,6	549,4	4,86			185,2	468,3	4,5			167,2	523,1	3,78			225,5	341,5	7,95
Piloto 3				274,5	623,3	5,55			168,4	478	4,34			298,7	684,5	4,43			309,9	508,7	9,48
				257,6	623,3	5,55			168,4	478	4,5			148,8	684,5	4,43			208,8	508,7	9,48

Best distance	148,8	Plane 3
Best lenght	684,5	Plane 3
Best time	9,48	Plane 4

Best plane: No. 3

Figure 10. This table condenses Nadia's calculations through which she concluded that plane 3 is the best one.

By applying this model, Nadia concluded that the best plane was the number 3. This because the plane got the shortest distance from the target (148.8) and the longest length of throws (684.5).

In [15] Margarita reacted to Nadia's previous comment. She *got in contact* with Nadia by paying attention to the table included in the attachment. In her comment Margarita is also *identifying* new ways of addressing the problem:

[15]
Theme: Re: welcome and organisation
From: Margarita
Date: Wednesday, 2nd of April 2008, 01:34

Nadia: At the beginning your table looked fine to me since it corresponded with my main idea, but then there was something that made me take paper and pencil [...] I started to graph the points of each plane using Excel. We needed to relate both, the flight time and the length of throw. I remembered my probability lessons and how to find a linear regression. But before

> calculating it, I was looking at the graphs and I thought: What is needed in order to be the best plane? "To achieve the longest distance in the shortest time"....That is, the plane having the line with the biggest slope will be the winner. It is clear that by calculating the slope we are finding the quotient of distance and time, and this is the velocity of the plane. Therefore the linear regression will allow us to calculate the slope […]
>
> As you will see in the first four graphs, one for each plane, I differentiated the pilots but this does not cause a significant change in the location of the points.
>
> […] When selecting the best plane, should we just take into consideration the velocity of the plane? Should we use the slope of the line? If yes, then the graphical data are consistent with your answer Nadia: THE PLANE 3 IS THE BEST ONE
>
> In this analysis I did not considered the position where the plane landed, nor the distance from target [...]
>
> Margarita

Margarita proposes in [15] another way to find the best plane of the contest, which is based on the assumption that the best plane will be the one achieving the longest distance in the shortest time. As in the case of Nadia's comment [14], the utterance [15] was complemented with an attached file, which includes the linear regressions used by Margarita to declare plane number 3 as the best one (see figure 11). It is important to notice here the differences between the two strategies suggested by Nadia and Margarita. While Nadia is considering all the variables involved in the problem (distance from target, length of throw, flight time and pilots), Margarita is only taking into consideration the length of throw and the flight time (figure 11). Margarita decided to discard the pilots because their performance did not affect significantly the slope of the linear regressions she calculated (see figure 12).

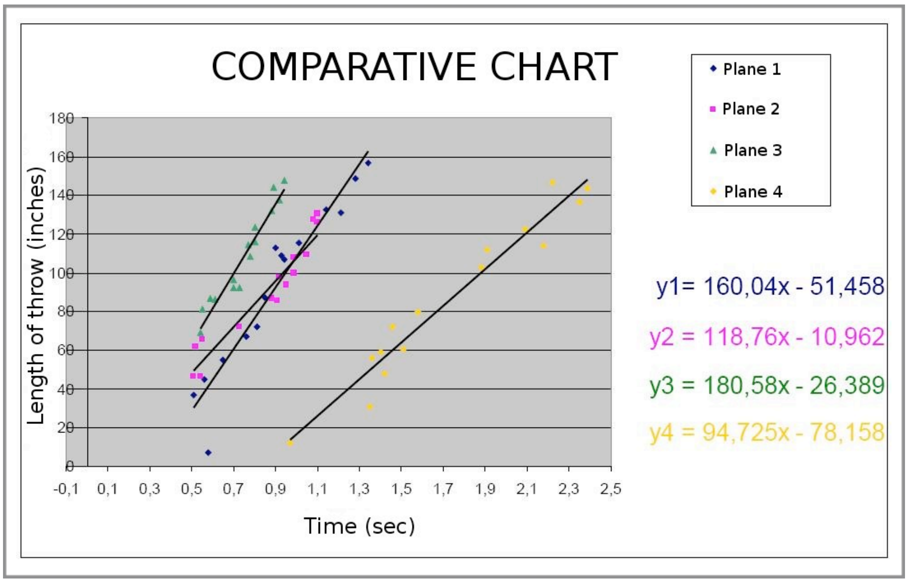

Figure 11. Graph containing four linear regressions relating the length of throw and the flight time for each of the planes. In this case pilot's performance is not considered.

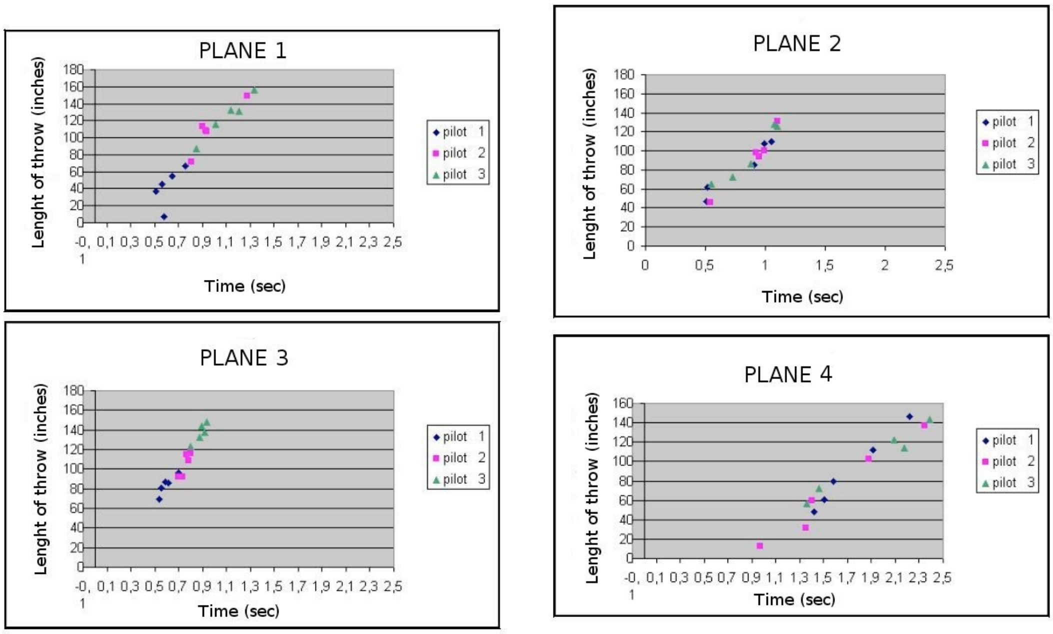

Figure 12. This figure shows four linear regressions relating the length of throw and the flight time for each of the planes in which pilot's performance is differentiated.

Some hours after, Nadia returned to the forum with a new idea about the mathematical task:

[16]
Theme: Re: welcome and organisation
From: Nadia
Date: Wednesday, 2nd of April 2008, 12:18

[…] Margarita, what you thought and did with the slopes of the regression lines was perfect! Excellent way! [...]

You know? I have been thinking on the problem from a vector point of view […] And I came up with some statistics for the shots, because we got the coordinates of the landing points. But every time I have more questions (everything is in the attachment […])

Nadia

Nadia is *keeping the contact* with Margarita by referring to her proposal of using the slopes of the linear regressions to select the best plane of the contest. Nevertheless, in her comment Nadia is *identifying* a new way of selecting the winner of the contest. She continues using all the variables included in the problem, but in this case she applies measures of statistical dispersion. Nadia used particularly:

- The arithmetic mean $(\bar{x})$
- The standard deviation (σ)
- And the coefficient of variation $\left(C_v = \frac{\sigma}{\bar{x}}\right)$

The way she used these measures is illustrated in figure 13. This figure shows some of the calculations that Nadia included in her attachment and in which she focuses on the variable "distance from target". The figure includes twelve small tables arranged in three rows and four columns. The rows represent the pilots, while the columns represent the planes.

Pilot 1 Plane 1			Pilot 1 Plane 2			Pilot 1 Plane 3			Pilot 1 Plane 4			Mean pilot	SD pilot	CV pilot
x1	y1	Distance	x1	y1	Distance	x1	y1	Distance	x1	y1	Distance			
45	-78	90,0	-37	-30	47,6	-42	-4	42,2	15	66	67,7			
7	-78	78,3	-48	-9	48,8	45	5	45,3	32	-46	56,0			
55	-42	69,2	28	26	38,2	35	-7	35,7	5	32	32,4			
-14	-46	48,1	12	-35	37,0	-13	5	13,9	14	-21	25,2			
21	-29	35,8	-19	26	32,2	4	-11	11,7	27	-5	27,5			
22,8	-54,6	64,3	-12,8	-4,4	40,8	5,8	-2,4	29,8	18,6	5,2	41,8	44,1467718	14,4896876	0,32821624
Pilot 2 Plane 1			Pilot 2 Plane 2			Pilot 2 Plane 3			Pilot 2 Plane 4					
x2	y2	Distance	x2	y2	Distance	x2	y2	Distance	x2	y2	Distance			
-12	26	28,6	-11	13	17,0	-4	12	12,6	-13	22	25,6			
-40	-20	44,7	-14	-36	38,6	24	9	25,6	-3	-21	21,2			
-38	22	43,9	31	13	33,6	-9	34	35,2	-11	-51	52,2			
-61	15	62,8	-43	10	44,1	-39	29	48,6	-12	56	57,3			
-48	61	77,6	-14	50	51,9	40	21	45,2	5	-69	69,2			
-39,8	20,8	51,5	-10,2	10,0	37,1	2,4	21,0	33,4	-6,8	-12,6	45,1	41,7833939	8,12010278	0,19433804
Pilot 3 Plane 1			Pilot 3 Plane 2			Pilot 3 Plane 3			Pilot 3 Plane 4					
x3	y3	Distance	x3	y3	Distance	x3	y3	Distance	x3	y3	Distance			
42	71	82,5	-9	6	10,8	24	41	47,5	53	-62	81,6			
61	38	71,9	-27	-13	30,0	-3	52	52,1	52	54	75,0			
43	27	50,8	17	-17	24,0	-31	54	62,3	48	-26	54,6			
18	50	53,1	14	47	49,0	5	64	64,2	45	25	51,5			
15	6	16,2	-36	41	54,6	36	63	72,6	25	40	47,2			
35,8	38,4	54,9	-8,2	12,8	33,7	6,2	54,8	59,7	44,6	6,2	62,0	52,5622039	12,9256048	0,24591063
Mean plane		56,9			37,2			41,0			49,6			
SD plane		6,61063			3,54677			16,3399			10,8286			
CV plane		0,11617			0,0954			0,39877			0,21833			

Figure 13. In this figure different values of arithmetic means, standard deviations (SD) and coefficients of variation (CV) are shown. The values correspond with the "distance from target" measures that each pilot and plane got during the contest.

I will explain Nadia's calculations starting with the small table located in the upper left corner of figure 13. This table represents the distances from target that pilot 1 scored during his five throws made with the plane number 1. Nadia calculated the arithmetic mean of the five distances recorded (90, 78.3, 69.2, 48.1 and 35.8) and as a result she obtained 64.3. Nadia applied the same procedure in the 11 remaining tables, getting a total of 12 arithmetic means.

Next, Nadia focused on the arithmetic means obtained in each row and each column. Take for example the first row (from top to bottom), which represents pilot 1. Here Nadia considered the set of values 64.3, 40.8, 29.8 and 41.8, and calculated the arithmetic mean (44.14), the standard deviation (14.48) and the coefficient of variation (0.32). If we now consider the first column (from left to right), which represents plane 1, we see that

Nadia used the values 64.3, 51.5 and 54.9 to obtain the arithmetic mean (56.9), the standard deviation (6.6) and the coefficient of variation (0.11). Finally, Nadia used the coefficients of variation to identify the best pilot and the best plane. She considers smaller coefficients of variation to be better than the larger ones; therefore she declared pilot 2 ($C_v = 0.19$) and plane 2 ($C_v = 0.09$) as winners of this category. It is not clear from the attachments why she concluded this, since a small coefficient of variation does not necessarily implies having a small deviation from target. Take as an illustration an extreme case where a plane landed 15 times at a distance of ten meters from the target. This would be an example of a plane landing away from the target, but having a coefficient of variation equal to zero.

In two subsequent utterances that were published the same day (not included in this analysis[30]) Nadia presented similar calculations for the variables "length of throw" and "flight time". In each of them, Nadia got different winning planes. I think this situation placed Nadia in a state of perplexity or confusion since in one of her attachments she stated: "What should I look at? Which one is the best plane? Which one is the best pilot?" Margarita reacted in this way to Nadia's calculations and questions:

[17]
Theme: Re: welcome and organisation
From: Margarita
Date: Wednesday, 2nd of April 2008, 13:09

[…] I read your questions and your tables […] I don't think we have to choose the best pilot. The activity asks: "to judge what is the best paper airplane". They use different pilots to avoid the dependence of the final result on the

[30] I did not include these utterances, neither one published after the utterance [23], because their text was too short and did not provide any idea for the discussion. The ideas the teachers wanted to communicate were mainly contained in files attached to the utterances. Therefore I only considered these attachments in the analysis.

> throwing ability of a particular person. Therefore I think we should focus on finding the easiest way to handle the data and choosing the best plane. What do you think?
>
> Margarita

Margarita is *keeping the contact* with Nadia, but her utterance [17] can be also considered as an *evaluative* act since she suggests to Nadia to disregard the pilots in the analysis. Margarita also suggested finding an easier way to select the best plane.

I argue here that the utterance [17] from Margarita, together with the different results produced by the model presented en [16], made Nadia to experience a *mathematical reflection*. More precisely, I think Nadia *located* the need of passing through a *systematization* stage of the problem before starting its *mathematization*[31]. In the systematization stage the way to determine the best plane should be defined. This interpretation is supported by the fact that three days after her participation in [16], Nadia returned to the forum with the following comments that I interpret as the outcome of her mathematical reflection:

> [18]
>
> *Theme:* Some issues
>
> *From:* Nadia
>
> *Date:* Saturday, 5th of April 2008, 06:17
>
> […]
>
> Something does not make sense to me: that a plane has flown further means that the plane will reduce its proximity to the target, therefore I would not consider jointly the three criteria to evaluate the planes. Obviously the one flying further and staying more time on the air will not hit the target (0, 0), even if the pilot had a good shot […] We were proceeding in a way in which we did not know what we were looking for, neither how the performance of the planes was.

[31] See figure 6, chapter 4.

Now: We could choose the ten shots that are closer to the target, and then from those, to choose the one that made it in more time and with the longest, what do you think? […]

Nadia

[19]

Theme: Re: Some issues
From: Nadia
Date: Saturday, 5th of April 2008, 06:26

…

or maybe, among the ten airplanes that flew more time, to choose the one that was closer to the average value of the distance from target.

Or among the ten planes that flew more time, to choose the three planes that travelled a longer distance, and from those to pick the one which got closer, or...let's establish the best criterion, let's think together...ok? […]

Nadia

The phrase "We were proceeding in a way in which we did not what we were looking for" from [18] is considered as evidence suggesting that Nadia *located* the need to define how to determine the best plane before starting the mathematization of the modelling task. I have argued that the utterance [17] where Margarita suggested disregarding the pilots may have influenced the mathematical reflection of Nadia. I argue that because, although Nadia does not explicitly refer to this aspect on [18] or [19], Nadia stopped focusing on pilot's performance during her calculations. Now that Nadia and Margarita have (implicitly) agreed to eliminate pilot's performance from their models, they keep looking for a model that allows them to determine the best plane of the competition. Nadia offers some suggestions about this in [18] and [19] that are taken up by Margarita in [20]:

[20]

Theme: Re: Some issues

From: Margarita

Date: Saturday, 5th of April 2008, 21:44

As Nadia says, the data presented in the tables are too many and they cannot be, in my view, taken all into consideration...because what the best plane is? the one flying more time? the one flying further? the one getting closer to the target?

So I think it is important that we decide today a final criterion and thereafter write the letter we have to submit.

I agree with you in choosing the ten shots that are closer to the target and from that to choose the one that was faster. In this way the variables will be reduced to two […] But now I am thinking, why should we choose 10 shots? I suggest to do it in a different way, let's select the planes landing in a circle with centre (0, 0) and a fixed radio, and then select the one that made it in less time or in more time as you suggest […] but, do we judge the fastest one or the one staying longer on the air?...we can judge both aspects in a contest therefore we need to decide what to judge. What do you think?

In a model we need to take into consideration some aspects and disregard others, because it is a model. What should be the size of the radio? Or should we take a fixed amount of shots? I think the idea of the radio is similar to that of the shooting competitions like archery.

Margarita

Margarita is *keeping the contact* with Nadia in [20]. She supports the idea of considering fewer variables in the model, but she queries Nadia's suggestion regarding the selection of ten shots. This can be regarded as an *evaluative* act. In [20] Margarita *advocates* an alternative model, which considers a circle with a fixed radius instead of considering 10 shots. She argues that this is more similar to what happens in the shooting competitions. In [21] Margarita crystallises or *identifies* her proposal of the circle with a fixed radio:

[21]
Theme: Re: Some issues
From: Margarita
Date: Saturday, 5th of April 2008, 22:32

Colleagues: I am writing you because I think 20 could be a good size for the radius, since it is one fourth of the distance from the starting point to the target point. In this way we get six shots with three planes, I mean, the fourth plane was not involved since it did not surpass the first filter. [T]hen we can evaluate the next aspect... if we calculate the maximum velocity (of course, calculating it by using the values of each shot)

What do you think? [...]

Figure 14 shows a table that has been extracted from a file attached to Margarita's comment in [21]. This table illustrates the evaluation procedure *advocated* by Margarita. Firstly, she located the planes whose distance from target was less than 20. The planes 1, 2 and 3 meet this criterion. Of these planes the winner is the plane that has flown with a maximum velocity. The velocity is calculated by dividing the value "length of throw" by "flight time". For example, the winner plane according to this model is number 3 because the plane reached a maximum average speed of 140.98 in/sec (which is obtained when dividing 86 inches by 0.61 seconds).

	POSITION		DISTANCE FROM TARGET (inches)	LENGHT OF THROW (inches)	FLIGHT TIME (seconds)		Within a circle of radius 20 inches	Velocity of the shot
	X	Y						
plane 1	15	6	16,2	87,3	0,85		accepted shot	102,71
plane 2	-11	13	17	93,6	0,95		accepted shot	98,53
plane 2	-9	6	10,8	86,5	0,88		accepted shot	98,30
plane 3	-13	5	13,9	86	0,61		accepted shot	**140,98**
plane 3	4	-11	11,7	69,1	0,54		accepted shot	127,96
plane 3	-4	12	12,6	92,1	0,73		accepted shot	126,16

Figure 14. This table shows the planes that landed within a circle with centre (0, 0) and a radius of 20 inches. Of these planes the fastest was the plane number 3 since it flew with a velocity of 140.98 in/sec.

The following was the reaction of Nadia to Margarita's proposal expressed in [21]:

[22]

Theme: Re: Some issues
From: Nadia
Date: Sunday, 6th of April 2008, 12:19

Girls, Margarita: The radius you mention is somewhat arbitrary, why we do not take advantage of the fact that we already have the arithmetic mean of the distances from target and consider the planes that were above that mean??? […]

[23]

Theme: Re: Some issues
From: Nadia
Date: Sunday, 6th of April 2008, 13:03

Well, here you have what I did according to the previous observation about the radius. But I want to mention that I also like your conclusions Margarita.

If you agree, vote, choose one of the three options, or choose them all because for me all of them are right. I mean, all are equally valuable and correct. There are as many answers as ways of evaluating [...]

I interpret [22] as an *evaluative* act in which Nadia describes as arbitrary the choice of the radio size suggested by Margarita. Nadia in turn *advocates* a counterproposal for finding the best plane. This counterproposal was crystallised or identified in an attachment included in [23]. The counterproposal consisted in considering all the measures collected in the contest[32] and then to calculate the arithmetic mean of the variables "distance from target", "length of throw" and "flight time". In this case the values of the arithmetic means are 46.2, 94.4 and 1.1 (see the small table to the right on figure 15). After this, Nadia applied two filters: (1) she identified the ten shots closest to the target, and (2), from those, she selected the planes whose flight times were equal or above the arithmetic mean (1.1 seconds). The only plane that met these criteria was the plane number 4 (see figure 15).

In a comment posted after [23] (not included in this analysis), Nadia *identified* a variation of this model. In this case she also used the arithmetic mean of the three variables and applied two filters: (1) she identified the ten shots closest to the target; and (2), from those, she selected the planes whose lengths of throw were equal or above the arithmetic mean (94.4 inches). The only plane that met these criteria was again the plane 4.

[32] Here I refer to all the measures included in table 4, chapter 4.

Pilot	Flight	Plane	Distance from target	Lenght of throw	Flight time
1	4	4	25,2	60,6	1,51
2	1	4	25,6	102,8	1,88
2	2	4	21,2	59,1	1,4

Distance from target	Lenght of throw	Flight time
46,2	94,4	1,1

Figure 15. Results of the model proposed by Nadia in which the planes that are closer to the target and have a flight time equal or above the average (1.1 seconds) are selected. According to this model the plane 4 is the best one.

I also want to point out that the second paragraph of the utterance [23] can be interpreted as the product of a *mathematical reflection*. When Nadia says, "choose one of the three options, or choose them all because for me all of them are right. I mean, all are equally valuable and correct. There are as many answers as ways of evaluating [...]", my interpretation is that, after considering the different criteria for selecting the winning plane that she and Margarita have got during their interaction, she has *located* a very important feature of a modelling process, namely, the many and different responses that can be obtained for the same problem under consideration.

Margarita in turn issued the following response to Nadia's counterproposal:

[24]

Theme: Re: Some issues
From: Margarita
Date: Sunday, 6th of April 2008, 20:22

Nadia, I have looked at your last two contributions and I agree on taking a longer radius and not 20 as I suggested...but anyway (all are arbitrary), but in both cases the best plane is the number 4 and in all the previous drafts, yours and mine, we coincided with declaring plane 3 as the best one. Therefore we are in big trouble...because […] where is the justification to dismiss plane 3 and choose the number 4[?]

We are in big trouble...and the time is running out.
Margarita

The utterance [24] could be interpreted as an *evaluative* act in the sense that Margarita also describes as arbitrary Nadia's choice of the radius. However, I also interpret [24] as a *challenging* act because Margarita is pushing the discussion in a new direction when she asks for the justification for selecting one plane (or applying one model) and not the other. Nadia reacted in this way to the *challenging* act:

[25]
Theme: Re: Some issues
From: Nadia
Date: Sunday, 6th of April 2008, 20:51

Increase the radius, make a small adjustment Ms. Judge, ha ha ha. Be good and give a hand to No. 4... why are you so rigorous? Anyway the choice is arbitrary. Don't you think?

I also interpret [25] as a *challenging* act, since Nadia is suggesting to benefit plane 4 without providing any justification, as Margarita requested. The challenge consists on trying to persuade Margarita to forget about the need of a justification. This is Margarita's reaction to [25]:

[26]
Theme: Re: Some issues
From: Margarita
Date: Sunday, 6th of April 2008, 21:47

The issue is what if the owner of the plane 3 shows up[?], what criteria would we use to justify that we did not use the drafts where that plane was the winner but we used the other one[?]. On top of that, remember that following different paths we found that the winner was the number 3 [...]

[26] is interpreted as the outcome of a *extra-mathematical reflection*. It is extra-mathematical because in her utterance she is not focusing on the mathematics involved in the modelling process itself, but on the consequences that the application of a particular mathematical model can produce. Particularly, Margarita in [26] is addressing the issue of *responsibility* in mathematically based decision making. This element is characterised in Alrø & Skovsmose (2002) in the following way:

> "It is clear that responsibility does not simply mean checking the mathematical one extra time. Responsibility in this situation includes something different. It presupposes an understanding of the context in which the mathematically based decisions are made. How do the calculations support making a certain decision? Could the decision be justified or questioned for other reasons?" (p. 217).

Unfortunately the dialogue between Nadia and Margarita was interrupted due to the lack of time. The end of this asynchronous forum was scheduled for April 6, 2008 at 24:00 hours and, as we have seen in [26], this discussion remained active until a few hours before the end of the forum. At this point Nadia and Margarita have not reached a consensus regarding the model that should be selected. This situation apparently made them to interrupt the discussion and in a message posted by Margarita on April 6, 2008 at 23:38 hours, she proposed to adopt the model suggested by Nadia in [23] and to declare plane number 4 the winner.

5.4 Results

In Section 5.3, I have shown the analysis of the interactions within three different working groups. The groups are different because in two of them reflections appeared, while in the third one they were not manifested. The analysis of such interactions consisted of characterising, in terms of the communicative characteristics of the IC-Model, the type of interactions of each group. The results indicate that there are differences and similarities in the communicative characteristics of the interactions. To discuss these differences and similarities I will use the following table, which is a graphical representation of the communicative characteristics that were present in each of the interactions.

Cases / **Communicative characteristics**	Case 1 (Reflection)	Case 2 (No reflection)	Case 3 (Reflection)
Thinking aloud	■	■	■
Getting in contact	■	■	■
Identifying		■	■
Locating	■		■
Reformulating			
Advocating			■
Challenging			■
Evaluating	■		■

Table 5. The shaded boxes in this table indicate the presence of a communicative characteristic within the considered interaction.

I will begin by discussing the communicative similarities among the three interactions. One similarity is the presence of *thinking aloud* acts. However, thinking aloud acts are indispensable elements to initiate a process of inquiry. It is not possible to initiate the collective resolution and discussion of a mathematical activity without having someone, at some moment, publicly expressing her/his ideas and thoughts about the activity.

Thinking aloud acts are sort of starting points in a process of collective inquiry. What I want to point out here is that the presence of this communicative act in the three interactions seems natural and not decisive for the emergence of reflections.

Table 5 shows that *getting in contact* was another communicative feature present in the three interactions. However, it is important to note that the *quality of the contact* was not the same in all the interactions. In the case 2, where no reflections appeared, the contact was quite unidirectional. I mean, it seems that Sandra was paying attention to Juan's ideas and suggestions, while Juan ignored Sandra's ideas in more than one occasion (see [9] and [14]). In the cases 1 and 3, where some reflections were manifested, the contact was made by more than one person. The case 1 shows that Susana and Mariana always kept the contact with Alberto: they were always listening to and evaluating Alberto's ideas (see [2] and [3]). In the case 3 the contact between Nadia and Margarita was reciprocal. During all the interaction both teachers were listening to and analysing their proposals to select the best plane (see [15], [16], [17] and [20]).

Now I will focus on the similarities between the cases 1 and 3, in which some reflections were identified. I do this in order to try to identify the common characteristics between these two interactions that could be interpreted as factors favouring the emergence of reflections. In addition to the *thinking aloud* and *getting in contact* acts that I already mentioned, these two interactions had in common *locating* and *evaluating* acts.

The presence of *locating* acts in both interactions has an explanation that is related to my interpretation of a locating act. As mentioned at the beginning of this chapter, I decided to label as locating acts the moments of an interaction in which a teacher discovered something new about the topic being discussed. My interpretation of a locating act was based on the following explanation from Alrø & Skovsmose (2002). According to them to locate means: “finding out something that you did not know or was not aware of before” (p. 101). The reader will notice that the locating acts coincide with the utterances that I have identifying as outcomes of a reflection (see for example [4], [18] and [23]).

The *evaluative* acts were another element common to the interactions where reflections appeared. And also, an element that was not present in the example 2, where there were no interactions. I claim that the evaluative acts were crucial for the emergence of reflections.

For instance, in the case 1 the evaluative acts of Susana (see [2]) and Mariana (see [3]), were a sort of trigger that pushed Alberto to revisit his ideas about the interpretation of graph 5. If these evaluative acts had not been present, probably Alberto would not have doubted of his own interpretation, nor have felt the need to revise his mathematical ideas underlying such interpretation.

Another example is the case 3. I already have mentioned that the evaluative act expressed by Margarita in [17], where she suggests to disregard pilot’s performance, may have contributed to the emergence of the mathematical reflection experienced by Nadia (see [18]).

Another communicative characteristics that I consider as driving forces for the emergence of reflections are the c*hallenging* acts. I am interpreting a challenging act as “the attempt to push things in a new direction or to question already gained knowledge or fixed perspectives” (Alrø & Skovsmose, 2002, p. 109). For example, I think that the challenging acts expressed by Margarita and Nadia in [24] and [25] formed a base that allowed that the extra-mathematical reflection expressed in [26] took place. If Margarita had not changed the focus of the discussion in [24] towards the issue of justification of the selected model, then it would have been more difficult to produce such extra-mathematical reflection.

Thus, my analysis suggests that there are communicative differences between the kind of interactions where reflections appear and those in which they do not appear. The main differences that have been detected in the analysed interactions are:

- There are differences in the *quality of the contact* established between the participants of an interaction. The type of contact present in interactions where reflections appeared seems to be richer and diverse, that is, the contact tends to be reciprocal and to be established by more than one person.
- The interactions in which reflections have emerged have a greater number of *evaluative* and *challenging* acts. These two elements seem to be necessary for the emergence of reflections.

In relation to the point (2) above-mentioned, I want to clarify that I am not suggesting that the evaluative and challenging are sufficient ingredients for the appearance of a reflection. I think people receiving evaluative and challenging acts on their actions and ideas may react differently to such inputs. I would say for instance that Alberto (case 1) reacted in a very positive way to the evaluative acts from his colleagues. He decided to revise his mathematical interpretations, and afterwards he publicly expressed a change of opinion on the matter. Nevertheless, I think there are people who are more sensitive, that could react in a different way to this sort of evaluations. There may be people who feel attacked or depressed by such comments, without experiencing any reflection. My point here is that, in order to be benefited from this kind of communicative acts, it is probably necessary to have people willing to share their ideas and make them subject to public inspection.

5.5 Discussion of the Inquiry Co-operation Model

In this last section of the chapter I would like to reflect on the potentiality and limitations that I identified when I used the IC-Model as a tool for characterising online interactions.

The first advantage I will mention is its range of applicability. Although the IC-Model was developed through the observation of face-to-face interactions between mathematics students and their teachers, it was possible to apply it in the online setting where I developed my research. There are two properties of the model that facilitated its application in an online setting:

1. The communicative characteristics that define the IC-Model are not medium-dependent. This means that these communicative characteristics can be expressed and identified in both, verbal and written communication, as well as in the cases of synchronous and asynchronous communication.
2. The communicative characteristics that define the IC-Model are not subject-dependent. The communicative characteristics are a means to characterise human interactions regardless of the type of “students” and “teachers” who are involved in the interaction. For instance, in this research mathematics teachers involved in an in-service course played the role of the students.

Nevertheless, the application of the IC-Model in the analysis of the data was not straightforward. It was necessary to contextualise and rethink the communicative characteristics of the IC-Model within an online setting.

Initially I had difficulties distinguishing communicative acts in the interactions. For example, how to identify in an asynchronous forum a person who had *located* a particular idea? According to Alrø & Skovsmose (2002) a *locating* act is related to the process of discovering possibilities and finding something new (see p. 14 and p. 101). Therefore I labelled as locating acts those utterances where people somehow expressed that had discovered something new. Such was the case of Alberto in [4]. This utterance shows that Alberto has discovered that his interpretation of the graph 5 (see figure 7) was incorrect. However, his explanation is strongly supported by mathematical elements… Could then be [4] considered as an *identifying* act where Alberto is crystallising and making visible to the group his mathematical ideas? It was necessary to establish clearer criteria for differentiating communicative acts and to try to avoid such ambiguities. In this case I kept the criterion of considering as *locating* acts those utterances where some sort of discovery was expressed by the teachers, regardless of whether they expressed it in mathematical terms or not. Those utterances where only mathematical ideas where expressed and clarified, but without expressing any kind of discovery, were classified as *identifying* acts.

Another aspect of the contextualisation of the IC-Model in an online setting is the influence of technological elements in the communicative acts of the model. Alrø & Skovsmose (2002) recognise that computer use in the face-to-face interactions may provide new ways of *thinking aloud* because it helps to make visible mathematical procedures (see p. 108). In an online setting, not only thinking aloud acts become visible and are shaped by technological elements. For instance, the files attached to utterances [15] and [16] (see figures 9, 10 and 11) show how the mathematical ideas that have been *identified* in the process of the interaction, can be materialised and communicated by means of graphical representations and numerical tables.

My point with these comments is to highlight the need to further investigate how the communicative characteristics of the IC-Model are contextualised, modified and become operational in an online educational setting.

A virtue of the IC-Model is that its application reduces the complexity of the study of interactions in an online setting. This reduction in complexity is achieved by reducing the focus of the researcher. The IC-Model makes you to focus solely on human interactions in an online setting, but particularly in the communicative characteristics of such interactions. When you use the IC-Model to analyse human interactions connected to a process of inquiry, what you get is a characterisation of such interactions. In my research this characterisation helped me to establish connections between the emergence of reflections and the communicative components of an online interaction.

However, the fact that the IC-model makes you to focus on the study human interactions makes it harder to detect possible relationships between the emergence of reflections and the interaction with non-human elements of the online setting. For example, when I analysed the interactions of this and other previously applied online courses, I noticed that teachers' ideas and actions are not only influenced by the ideas and comments of their colleagues, but they are also shaped by the influence of non-human elements. Examples of this influence are the graphical and numerical information that a teacher can get through the manipulation of data in an Excel file. Such information can influence how a mathematical problem is conceived or addressed. Another example is the complementary sources of information, such as web pages and books. Teachers can make use of these sources of information and influence and enrich their views and ideas in a discussion for example.

My point here is that, by reducing the focus to the study of human interactions, the IC-Model does not allow me to study the relationships between the emergence of reflections and the interaction with non-human elements that seemed to influence teacher's way of thinking. It was at this point that I began the search for a theoretical tool that could allow me to observe and study the possible influence of other kind of elements of an online course (not only human elements) on the emergence of reflections in mathematics teachers.

Thus, my research entered a second phase in which I designed and implemented a new online course for teachers, but I also used new theoretical elements to analyse the outcomes of that course. In the following two chapters of the dissertation I will present the elements that constitute this second phase of my research.

6. The second online course: Use of CAS

This chapter describes the contents of the second online course that I designed and applied as part of my research. I used this course to study the possible relationships between some of the non-human elements of an online course and the emergence of reflections. In the first part of the chapter the rationale behind the course and its general structure is discussed. In the second part a description of the particular activities that constituted the course is provided.

This chapter describes the second online course that I designed and applied. The *scientific aim* of this course was to help me to study the influence of the non-human elements of an online course in the emergence of reflections in mathematics teachers.

As already mentioned, when I use the term *non-human elements* I refer to the resources that a participant in an online course interact with, but which are intentionally provided by the teacher educator. These are resources that are part of the design of an online course. The resources can be of different nature: software, video, activities, articles, audio files, web pages. The two main characteristics of the non-human elements of an online course are: (1) they are elements that are intentionally provided by the course designer. The designer has control of them in the sense that he/she decides when and how they will appear within the course; and (2) they are elements that serve to represent and communicate mathematical ideas and didactical ideas that are considered relevant to mathematics teachers' development.

To study how the interaction with such non-human elements may influence the emergence of mathematics teachers' reflections, I used a blend of theoretical concepts that allowed me to: firstly, to design a course where the non-human elements were clearly located and their roles explicitly stated. Secondly, the theoretical concepts allowed me to make establish a connection between the set of non-human elements and the set of "effects" that such elements produced on mathematics teachers. In this chapter I will only refer to the theoretical concept I used to organise the non-human elements within the course, namely, the concept of *documentational orchestration*. The rest of the concepts that constitute the theoretical blend will be discussed in the next chapter.

The role of this chapter is to provide the reader with a description of the course content and structure. To describe this course, a similar structure to the one used in chapter 4 will be followed: in the first part of the chapter the general characteristics of the course are addressed; whereas in the second part the particular activities that integrated the course are discussed in more detail.

6.1 The rationale behind the course and its structure

This second online course was named "Technological innovations for mathematics teaching". The course lasted four weeks and it was applied during the months of November and December, 2008. The course was taken by the same group of teachers who participated in the modelling course (see chapter 4). After completing the activities of the course, teachers received six credits out of a total of 76 needed to get their master's degree in mathematics education.

As I mentioned in the introduction to this dissertation, since I was a master student in mathematics education, I have been interested in the use of technology (software, calculators, Internet) in the teaching of mathematics. My empathy for this kind of topic was one reason why I decided to design this course. However this was not the only motivation for this choice.

Another reason was that, academics and educational institutions in Latin America are increasingly interested in the use of technology in mathematics teaching. This it is a tendency whose existence I could confirm from three different angles, thanks to the professional experiences I had in the prior years to my doctoral studies. When I collaborated as co-editor of the proceedings of the RELME 18 conference[33] (see Lezama, Sánchez & Molina, 2005), I noticed that several teachers and researchers from different Latin American countries were conducting teaching experiments in their classrooms, using computers, calculators or even temperature and motion sensors. I got another perspective on this tendency through my relationship as academic advisor for the company Casio[34]. This position allowed me to perceive that there are several educational institutions in Latin America, public and private, that are investing in the purchase of technology (software, calculators, sensors) for the teaching of mathematics and science. Another experience that allowed me to confirm the existence of this trend was the national reform of secondary education that started in Mexico in 2005. I collaborated in this reform as co-author of textbooks (see Cantoral et al, 2006 and Cantoral et al, 2008), therefore I had to study the guidelines and requirements of this reform. During this process I discovered that this is the first educational reform in Mexico, where the Ministry of Education explicitly requires using technological elements (such as spreadsheets, dynamic geometry software and sensors) in the teaching of science and mathematics.

[33] RELME is one of the more important mathematics education conferences in Latin America. Researchers and mathematics teachers coming from this (and others) region of the world converge on this conference. More information about this educational event can be found in http://www.clame.org.mx/relme.htm

[34] Casio Computer Co., Ltd http://edu.casio.com/

This however is only a trend. The reader should not conclude that the use of technology in Latin America is well accepted by teachers and integrated into the mathematics curriculum. For example, Julie et al (2010) provide a description of the access and implementation of digital technologies in the teaching and learning of mathematics within several countries and regions. In the case of Latin America they assure:

> "In general, even under massive government implementation, there remain unequal access, unequal resources, and sporadic use of the digital technologies in schools […] the role of the teacher is very important, and his/her beliefs, insecurities and lack of mathematical and technical preparation affect the possible impact that the use in the classroom of these technologies can have on students' learning and even attitudes. The need for careful, considered and continuous work with teachers is thus extremely important" (p. 380).

They also claim:

> "In some of these [Latin American] countries (e.g. Uruguay) it also seems that mathematics teachers are still very resistant to change and to the inclusion of digital tools into their practice. In Argentina, Giuliano et al. (2006) observe that teachers have little knowledge of the possibilities offered by new technologies, and when they do use digital tools, they select their activities, contents and teaching strategies according to traditional teaching stances" (p. 373).

Latin America is experiencing a situation similar to that described in Artigue (1998). The integration of technology in mathematics teaching has an institutional legitimacy in the sense that several educational and governmental institutions approve it and advocate it. However, such integration has a limited educational legitimacy. This means that mathematics teachers are not fully convinced of the benefits that this integration will provide to their teaching. The teachers' resistance to integrate technological tools in mathematics teaching reported in Julie et al (2010) is a manifestation of such limited educational legitimacy.

Artigue (1998) argues that in order to overcome this resistance it is necessary to provide teachers with didactic tools that enable them to analyse how their practices are modified by the use of technology:

> "[S]uch resistant obstacles will not be overcome without giving didactic analysis a more important role in teacher training, and without providing teachers with didactic tools allowing them to analyse transpositive processes, to identify the didactic variables of situations and pilot them, and to analyse their professional techniques and the way these are modified by the use of computer technologies." (pp. 126 - 127).

Thus, another reason for designing this course was to contribute to the integration of technology in mathematics teaching through the didactical analysis of the transformations that the use of technology can produce in the mathematics classroom.

The *didactical aim* of the course was to make teachers aware of the potential changes that may occur in the mathematics classroom when the use of CAS[35] technology is introduced.

The courses on the use of technology which are usually offered in the CICATA program address the use of software with graphic capabilities, such as dynamic geometry software for example. I was interested in discussing with teachers the use of other technological tools, and the use of CAS seemed like a good option for two reasons. Firstly, the use of CAS may be relevant to the teachers enrolled in the educational program, regardless of the academic level in which they work. This is because CAS software can be applied in basic algebraic manipulations (such as those

[35] CAS is an acronym for *Computer Algebra System*. A software or a calculator with CAS permits to perform symbolic calculations with mathematical expressions.

discussed in basic algebra courses) or more advanced algebraic manipulations (such as those addressed in Calculus courses for example). Secondly, I had previously read the book Guin, Ruthven & Trouche (2005) and it seemed to me that some of the theoretical ideas and examples presented in the book would be relevant to the teachers participating in a course like this one.

The course was particularly focused on helping teachers to notice that: (1) new mathematical techniques may emerge, i.e., techniques that are only accessible through the use of technological tools, and (2) that some mathematical tasks and techniques could lose their meaning and become obsolete. I think these two points address and challenge traditional conceptions of mathematics teaching. In order to perceive them, teachers need to query the structure of their lessons, the exercises that they propose to their students and even their teaching methods.

Along the course the concepts of *tasks* and *techniques* were used in the sense of Chevallard (1999) (see section 6.2.1 for a brief illustration of these concepts). I used these concepts for structuring the course because I consider that they are useful to point out some of the transformations and didactical phenomena that may occur in a mathematics classroom when the use of a technological device is incorporated in the study of mathematics.

To illustrate the above-mentioned ideas (1) and (2), I set up a *documentational orchestration* (Sánchez, 2010a; Sánchez, to appear, b). It is difficult to provide a precise definition of this concept without introducing the concepts of *documentation work* and *documentational genesis* (these concepts will be introduced in the next chapter). However, at this point a documentational orchestration (DO) can be interpreted as a selection and arrangement of resources that a teacher educator carry out with the intention of establishing an interaction between a group of mathematics teachers and the arranged set of resources. Such interaction is aimed at promoting the professional development of the mathematics teachers.

A documentational orchestration is divided into stages. Each stage consists of several resources and it has a specific purpose.

The stages of the DO require the designer to make explicit the location and function of the resources. Such process gives order to the resources. When the set of resources is arranged in this way, it is easier to establish a connection between the set of ordered resources and the reflections that may emerge within the course. In other words, if a teacher's reflection appears during the application of a course it is easier to identify at which stage the reflection appeared and which resource triggered it.

Each of the stages that constitute the orchestration[36] presented in this chapter are represented in figure 16. The concepts of *task* and *technique* are two fundamental elements in the orchestration. The configuration of the orchestration lies in locating these two elements (tasks and techniques) in a lesson plan that has been designed for an educational setting based on the use of paper and pencil (stage 1). Later, teachers need to discuss the pertinence of such lesson plan in a setting in which the use of technology is allowed (stage 4). The discussion about the pertinence of the lesson plan should take place after two stages in which the teachers themselves have experienced some instrumented techniques[37] (stages 2 and 3). The role of these two stages is to raise awareness about the potentials and limitations of the instrumented techniques. The fifth and final stage is used as an

[36] From now on I will use the terms *course, orchestration* and *documentational orchestration* equivalently.

[37] These are techniques for solving mathematical tasks that are based on the use of technology. See Lagrange (2005) for a discussion of this sort of techniques.

institutionalisation phase, where the aim of all the previous stages is explicitly communicated to the teachers.

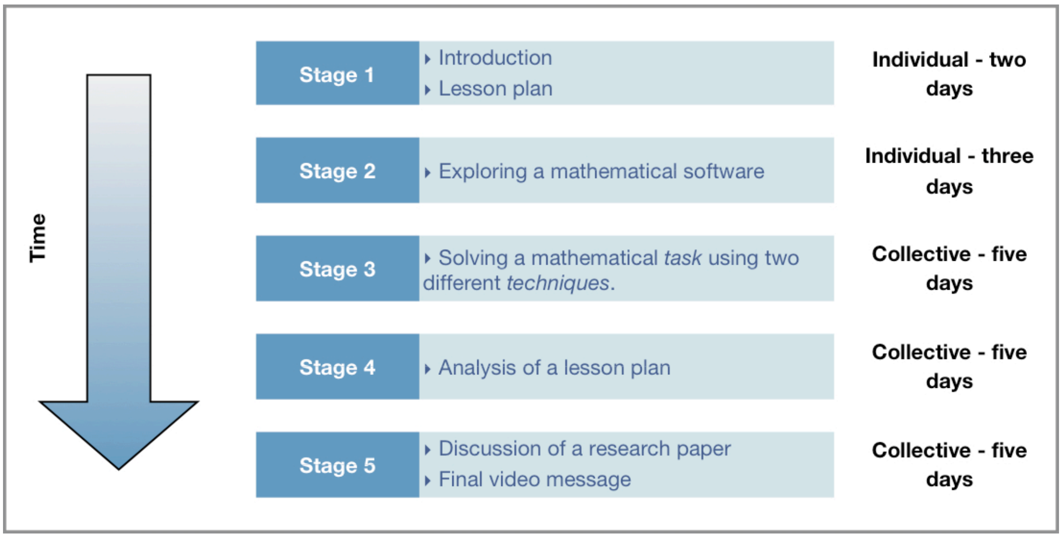

Figure 16. Graphical representation of the documentational orchestration. The orchestration consists of five stages, some of them are collective and other individual. The duration of each stage varies from two to five days.

6.2 The specific stages of the course

In this section a more detailed description of each of the stages of the orchestration is presented. The non-human elements that are part of each stage are specified.

6.2.1 Stage 1: Introducing the concepts of task and technique

During this stage the teachers were introduced to the course. The introduction was carried out through a document that the teachers could download from the online platform where the course was lodged. The document described the rationale behind the course, its aims, and how the performance of the teachers in the course would be assessed.

The aim of this stage was to introduce the teachers to the concepts of *task* and *technique*. Teachers were notified that the structure of the course was based on the concept of *praxeology* (Chevallard, 1999), and by means of an example the components of a praxeology were illustrated, namely: tasks, techniques, technology and theory.

The example describes a fictional situation where a high school teacher is presenting the topic "quadratic functions" to her students. Within the lesson the teacher presents to her students the ***task***: "find the roots of the real function $f(x)=x^2+x-6$". In order to help the students to solve this task, the teacher introduces a particular ***technique***, consisting in applying the quadratic formula:

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

In her introduction to the technique, the teacher explains how to interpret the terms a, b and c on the previous expression. She also shows, through some examples, that it is always possible to find the roots of any quadratic function by applying this formula. The discourse (the explanation, the examples) that the teacher uses to introduce and to illustrate the use of the technique is called ***technology***[38]. Thus, the students may successfully apply the taught technique, but probably they do not understand why the formula always works. The mathematical theory that explains and

[38] This is the only moment in the dissertation where I use the term *technology* in a praxeological sense. In the rest of the dissertation I use the word *technology* to refer to technological tools such as calculators, software, sensors and the Internet.

supports the operation of the technique is the fourth component of a praxeology and it is called the ***theory***.

This example is useful to illustrate the interdependence between the elements of a praxeology. That is, if one of its elements change, the rest of them will also undergo changes. For instance, suppose that the previous task is replaced by the new ***task***: "factorise the expression x^2+x-6 ". Now the ***technique*** consisting in applying the quadratic formula will not be enough. In this case, after obtaining the roots $x_1=-3$ and $x_2=2$, the students must replace them in the template $(x-x_1)(x-x_2)$ to finally obtain $x^2+x-6=(x+3)(x-2)$. As a consequence the ***technology*** or discourse used by the teacher should be modified. Besides explaining how the quadratic formula must be applied, the teacher should explain to her students why if $p(x_1)=0$ (where $p(x)=x^2+x-6$), then $(x-x_1)$ is a factor of $p(x)$.

The example was needed in order to introduce the first activity of the course. In such activity, the teachers are asked to locate a mathematical topic that they already have taught or that they like to teach. Afterwards teachers should produce a lesson plan for this mathematical topic. This is, they should identify the type of tasks and techniques that they usually present to their students when they introduce such mathematical topic. Finally, teachers should send their lesson plan by email to the person responsible for the course (myself).

The lesson plan helped me to verify that the teachers had grasped the concepts of task and technique, and that they were able to identify them within the structure of a lesson plan. I asked for this lesson plan at the beginning of the course to ensure that the contents of the course will not influence the structure of the plan. This is, I expected that most of the teachers would include tasks and techniques that are based on the use of paper-and-pencil[39] in the structure of their plans. As I will show in Section 6.2.4, the lesson plan also helped me to establish a context in which the teachers could discuss the relevance and validity of tasks and techniques that are based on the use of paper-and-pencil, in educational settings where the use of technology is allowed.

The two main non-human components with which teachers interact at this stage are the concepts of task and technique. The example that describes the lesson on quadratic functions serves to introduce these concepts, while the lesson plan serves to verify that the teachers have grasped their meaning and are able to identify the concepts within the structure of a mathematics lesson.

6.2.2 Stage 2: Acquainting teachers with the use of a software

To discuss with the teachers the effects that the use of technology may produce in mathematics teaching, it was necessary that the teachers themselves experienced the potential of a mathematical software. Thus, during the second stage of the course teachers were asked to solve a series of mathematical exercises with the help of the software *ClassPad Manager*[40]. It was decided to use this software in the course because it includes a CAS application. Therefore the software offered the opportunity of discussing with the teachers the potential changes that the use of CAS may produce on tasks and techniques in algebraic contexts.

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[39] I borrowed the term *paper-and-pencil* from Guin, Ruthven & Trouche (2005). In this work the term is used to denote mathematical tasks, techniques or even educational settings in which digital technologies such as calculators and software are not used. Only traditional tools such as blackboard, pencil and paper are utilised.

[40] See http://classpad.net/product/Classpad300/cp_manager_03.html

I am aware that there are other computer programs including CAS as one of its applications, however many of them are not free and this hampers their use and distribution. The ClassPad Manager software is not free either but, thanks to the support from the company that produces it, it was possible to provide teachers with a free copy of the software to use it during the course. The teachers could download a copy of this software from the online platform where the course was lodged.

The aim of this stage of the orchestration was that the teachers became familiar with the use of the software. To achieve this, teachers were asked to solve a list of mathematical exercises during three days. Some exercises required the use of software to draw graphs of functions, but most of the exercises required the application of CAS commands that would be used in the subsequent stages of the orchestration. An example of such type of exercises is the following one:

Apply the command "Factor" to the expression:

$$x^5 - 25x^4 + 216x^3 - 648x^2 - 432x + 3888$$

The exercises were complemented by video tutorials illustrating step by step how to use the software in order to solve the exercises (see figure 17). From previous experiences with face-to-face workshops for teachers on the use of technology, I noticed that several mathematics teachers (especially the older ones, I must say) have difficulties getting acquainted with the operation of advanced calculators and mathematical software. I expected such difficulties to appear at this stage of the course. Therefore I thought that the video tutorials would be a way of helping teachers to overcome such technical difficulties. The video tutorials illustrated step by step how to plot functions and systems of equations. They also showed different ways to enter and manipulate algebraic expressions, including one in which an equation editor is used to represent an algebraic expression in the software just as it is written on a blackboard or a textbook. The list of exercises as well as the video tutorials could be downloaded from the online platform. The video tutorial for the aforementioned exercise can be accessed through the link: http://j.mp/95UH3r

At this stage the CAS and the graphical capabilities of the software, the list of mathematical exercises, and the video tutorials were the non-human components with which teachers interacted.

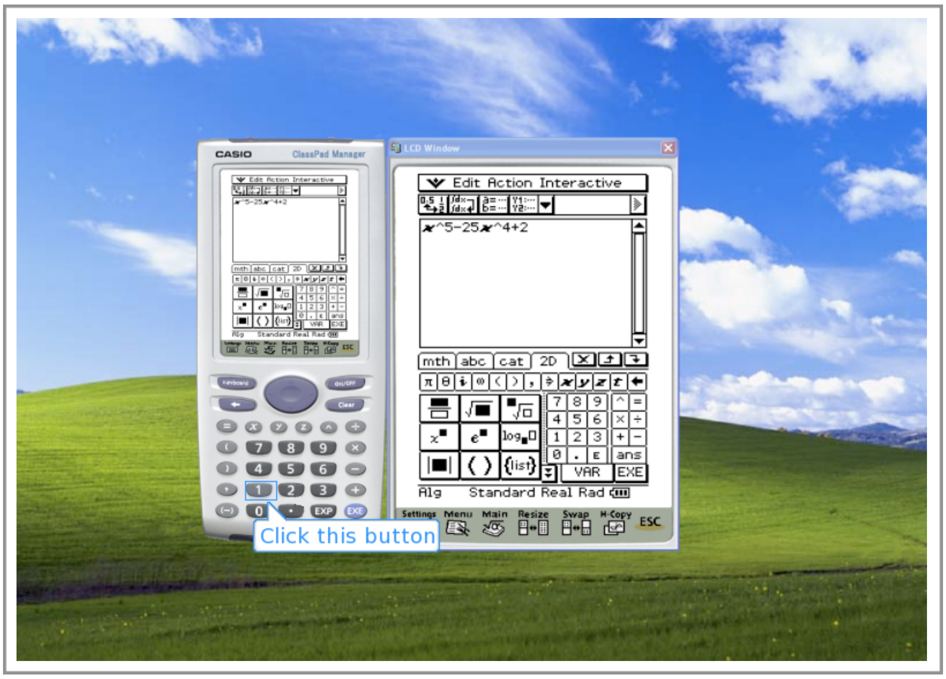

Figure 17. Screenshot of one of the video tutorials provided to the teachers. In the videos the way of introducing and manipulating a mathematical expression in the software is explained.

6.2.3 Stage 3: Solving a task with two different techniques

The third stage of the orchestration was inspired in the work of Mounier & Aldon (1996) presented in Lagrange (2005). Teachers were organised in teams of four or five members. Each of these teams was assigned to an asynchronous discussion forum, and there each team was asked to split into two sub-teams. Both sub-teams should work independently on finding a general factorisation for the expression $x^n - 1$, where $n \in \mathrm{N}$. With "general factorisation" I refer to a factorisation that is valid for different values of n. For instance, $x \cdot x^{n-1}$ is a general factorisation for x^n, where n can be any natural number.

To solve this task, one sub-team should only use paper-and-pencil, while the other one should only utilise the command *Factor* of the mathematical software. Such command was explored during the second stage of the orchestration. After solving the task, both sub-teams should meet and share their findings in the discussion forum.

According to the results shown by Mounier & Aldon (1996), it was expected that the sub-teams would obtain different results regarding the requested factorisation. On one hand it was assumed that the sub-teams working with paper-and-pencil would apply polynomial division and find that the expression $x^n - 1$ can be factored as $(x-1)\left(x^{n-1} + x^{n-2} + \cdots + x + 1\right)$. On the other hand it was expected that the sub-teams using the command Factor would find other possible cases of factorisation. Applying this command to the expression $x^n - 1$ they would notice that in some cases the obtained factorisation contains more than two factors (see table 6, chapter 7). It was even supposed that, in an effort to understand under what conditions the factorisation produce more than two factors, teachers would complement the Factor command technique with paper-and-pencil techniques, such as polynomial division.

The following are the necessary conditions for obtaining two or more factors in the factorisation of $x^n - 1$:

- The factorisation of $x^n - 1$ contains exactly two factors when n is prime. The two factors are $(x-1)$ and $(x^{n-1} + x^{n-2} + \cdots + x + 1)$.
- When n is a number bigger than two, the factorisation always produces more than two factors and $(x+1)$ is always one of them. Moreover, since the identity $x^2 - 1 = (x+1)(x-1)$ can be applied to the expression $x^n - 1$ when n is even, then the factor $(x-1)$ is also obtained in this case.
- When n is odd but not prime, the factorisation of $x^n - 1$ contains more than two factors: $(x-1)$ is one of them, but not $(x+1)$.

After sharing their results, the teachers should consider and answer the following *note of reflection*:

> "Assume the following situation: all of you are part of the mathematics academy of an upper secondary school. Among your responsibilities as academy members, you need to support and advice other mathematics teachers from the school.
>
> The school's mathematics curriculum has been recently modified due to an educational reform. The new curriculum allows the use of mathematical software with CAS capabilities for the study and teaching of mathematics, just like the software you have been used in the first part of this activity. One of the mathematics textbooks approved in the recent educational reform (and adopted by the school to which you belong), proposes the task of factorising $x^n - 1$ as an activity for the students. The textbook also indicates that to

solve this task, the teacher can introduce their students either to the 'Factor technique' or to a 'paper-and-pencil technique'. As members of the academy of mathematics,

1. Which technique would you recommend to a teacher who is introducing this particular activity in her classroom? Why?
2. What do you think would be the advantages (and disadvantages, if any) for the students when applying the mathematical technique recommended in 1?"

The teachers had to write a collective answer to these questions and deliver it by email. The purpose of this note of reflection was to trigger the comparison of instrumented and paper-and-pencil techniques in the factorisation of algebraic expressions.

The instrumented techniques and the paper and pencil techniques were the non-human components central to this stage. The overall purpose of this stage was that teachers could experienced both type of techniques and discuss their differences, advantages and disadvantages for the teacher (see question 1 in the note of reflection), and for the students (see question 2 in the note of reflection).

It was expected that among the advantages of the instrumented techniques, teachers would highlight their pragmatic value (Lagrange 2005). The ***pragmatic value*** of a technique refers to the efficiency and economy (of time, of effort) with which such technique helps to solve a mathematical task. For example, the pragmatic value of any CAS software may be related to speed and efficiency with which the software performs algebraic factorisations. However, it was also expected that the teachers (and particularly those who worked with the command Factor) would acknowledge some kind of epistemic value in the instrumented techniques. The ***epistemic value*** of a technique refers to its potential to serve as a means to understand the mathematical objects involved in the application of the technique. For instance, the epistemic value of CAS-based techniques may be related to the fact that such techniques allow a more experimental approach to elemental algebra, where through the use of software students can explore and produce conjectures, regarding the "effect" of the command Factor in the factorization of $x^n - 1$.

6.2.4 Stage 4: Analysing the pertinence of a lesson plan

The stages 4 and 1 of the orchestration are linked. During the fourth stage, one of the lesson plans that the teachers handed in during the first stage was selected. This lesson plan was distributed to the rest of the teachers to analyse it. The teacher who designed the selected lesson plan gave his consent to use it during this stage of the course. The selected lesson plan is the following[41]:

A LESSON PLAN
by Juan Castro

Mathematical topic: Solving a system of two linear equations in two variables.

The techniques: I provide the students with the following techniques:

- **Graphical solution of a system of linear equations:** This technique consists in drawing the graphs of the two equations on the same coordinate system, and determining the coordinates of the point where the graphs intersect.

[41] The teacher who designed this lesson plan also included in his description the *technology* (in the sense of Chevallard, 1999) that he uses to introduce the mathematical topic to his students. However, I will not include it in this presentation since I only find relevant to this discussion to present the techniques and tasks that the teacher included in his lesson plan.

- **Algebraic solution of a system of linear equations (Method 1):** An equation in a single variable could be obtained by a linear combination of the two original equations. The obtained equation is solved, and the value of the variable is substituted in any of the two equations originally presented. In this way you get the value of the second variable.
- **Algebraic solution of a system of linear equations (Method 2):** Solve one of the equations for one of the variables, and substitute this into the other equation. Now solve the resultant equation for one of the variables. Replace the obtained value of the variable in any of the two original equations and solve for the other variable.

The tasks: The tasks that I commonly propose to the students in relation to this topic are of the following type:

1. Find the solution of the following system using the graphical method. If there is not solution, illustrate it:

$$\begin{cases} y = 3x \\ x + y = 8 \end{cases}$$

2. Solve the following system using method 1:

$$\begin{cases} 2x - y = 0 \\ 2x + y = 4 \end{cases}$$

3. Using method 2, solve the next system:

$$\begin{cases} 3x + 2y = 11 \\ 5x - 4y = 11 \end{cases}$$

4. A board of 12 meters is cut into two parts, so that one of them is two meters longer than the other one. How long is each part of the board?

At this stage of the course teachers were again distributed into teams, and each of those teams was assigned to a discussion forum. The selected lesson plan was presented to each of the teams, and they were asked to discuss in the forum the possibility of applying this lesson plan in a classroom the use of technology is allowed. More precisely, they were asked to consider the following situation:

> "Think of a mathematics classroom where students have access to and know how to use a software (or a calculator) with algebraic and graphic capabilities just like those held by the software ClassPad Manager. Now focus your attention on Juan Castro's lesson plan. Pay attention particularly to the tasks and techniques presented there. If you apply the suggested lesson plan in such a classroom:
>
> 1. What would be the impact of the use of technology on the tasks?
> 2. What would be the impact of the use of technology on the techniques?
> 3. If in the point (i) or (ii) some sort of impact is reported, then: Do you think that such impact would have any consequence in students' mathematical learning?"

Again, teachers should produce a collective response to these questions and deliver it by email to the person in charge of the course.

Juan Castro's lesson plan was selected because it is based on the use of paper-and-pencil, but also because the elements *tasks* and *techniques* are clearly identified. It was expected that a lesson plan like this one, based on the use of paper-and-pencil, would make evident the need to implement some modifications before it could be applied in a technological-aided classroom. To reflect on such possible modifications was the goal of this stage of the orchestration. It was expected that the teachers would notice that some of the proposed techniques may become obsolete, since there are faster and more efficient instrumented techniques to solve the tasks. Something similar may happen to the tasks. At least the three first tasks would become meaningless, since the technology would help students to solve them just by pushing a couple of buttons on the keyboard of the computer/calculator. If teachers could perceive this, then it was also expected that they notice the need to redesign the lesson plan in order to implement it in the new setting.

One of the non-human elements with which teachers interacted at this stage was Juan Castro's lesson plan. However, the instrumented techniques experienced in the stages 2 and 3 also came into play at this stage.

6.2.5 Stage 5: Discussing a research paper and closing the course

The fifth stage was a moment of institutionalisation of the course content. The teachers and teacher educators who participated in the course discussed in an asynchronous forum the content of the article Lagrange (2005). Initially it was planned to focus the discussion on this article on the modifications in mathematical tasks and techniques reported by the author of the article. The intention was to compare the changes that the teachers may have detected in the fourth stage of the orchestration with those reported in the article. However, as discussed in the next chapter, there were changes in the focus of discussion of the article.

Additionally, when the aforementioned discussion forum concluded a video message was posted on YouTube. In the video message the purposes of each of the activities of the course were explicitly mentioned. The aim of the video was to clarify to the teachers the rationale behind each activity, and to officially close the course. If the reader is interested, the video can be accessed at: http://j.mp/aYKRAq

In this last stage teachers interacted with two non-human elements: the contents of the article by Lagrange (2005) and the video message hosted on YouTube.

In the next chapter the outcomes obtained after applying the course described in this chapter are presented.

7. Outcomes of the second online course

In this chapter some concepts provided by the documentational approach (Gueudet & Trouche, 2008a, 2009) are used to analyze the results of the implementation of the online course on the use of technology described in the previous chapter. The analysis focuses on identifying the instrumentalization and instrumentation processes manifested during the course. The application of these concepts proved to be useful to observe the influence of non-human elements in the emergence of teacher's reflections. One of the main results is that some of the theoretical concepts from mathematics education research have the potential to trigger the emergence of didactical reflections on mathematics teachers.

In this seventh chapter of the dissertation, the implementation of the course described in the chapter six is analysed. Such analysis is focused on observing the interactions between some of the non-human components of the course and the mathematics teachers who participated in it. I am interpreting the interaction between the teachers and the non-human elements of the course as the way in which teachers use and appropriated those elements, but I also include the influence that such non-human elements may exert on teachers' way of thinking and acting within the course.

The purpose of analysing this kind of interactions is to investigate whether or not the non-human elements of the course influenced the emergence of reflections in mathematics teachers, and if that happened, to try to clarify the nature of such influence.

In order to analyse this type of interactions, some of the theoretical concepts provided by the *documentational approach* (Gueudet & Trouche, 2008a, 2009) are applied. The concept of *documentational orchestration* (Sánchez, 2010a; Sánchez, to appear, b) also plays a role in the analysis of the data. A general description of the concept of documentational orchestration was introduced in the previous chapter. In this chapter a more precise definition is provided.

7.1 Introducing the documentational approach

All the above mentioned concepts are introduced in this section. After this introduction, the way in which the empirical data were sorted out and selected is briefly discussed. Then, the results obtained by applying the theoretical concepts in the analysis of the data are presented. The last part of the chapter includes a discussion of the implications of the results presented, and a brief reflection on the use of the documentational approach on this research.

7.1.1 On the concept of documentational genesis

In the research paper written by Gueudet & Trouche (2009), a way of "tracking" the professional development of mathematics teachers is proposed. To accomplish this, Ghislaine Gueudet and Luc Trouche suggest to focus our attention on the activities that mathematics teachers develop outside the classroom, but that influence their work within the classroom. The focus is particularly centred on *teachers' documentation work*. That is, the interaction between the teachers and a set of elements that allows them to shape and define their work in the classroom. Expressions of such interaction are for example: to extract examples and exercises from a textbook in order to include them in their lesson plans; to analyse their students' mathematical productions; to listen to the suggestions, ideas and experiences from colleagues; to review the contents of a website that contains educational materials; to study a curriculum reform to be applied in their own school, etc. The set of elements with which a teacher interacts during her documentation work is called *resources*. When a group of teachers is participating in a collective work project, where they share and interact with a common set of resources, then we can speak of a *collective documentation work* (Gueudet & Trouche 2008a; 2008b).

In this new approach it is claimed that, when an interaction between a teacher and a set of resources takes place, a *documentational genesis* (DG) may appear. The concept of DG can be interpreted as an analogy[42] of the concept of *instrumental genesis* (Rabardel 1995; Trouche 2005b) applied to the field of mathematics teacher education. Like the instrumental genesis, the DG is a two-way process in which the teacher appropriates and/or modify the set of resources with which she interacts (this part of the process is called instrumentalization), but the set of resources also shapes and influences teacher's activity and way of thinking (this part of the process is called instrumentation). Thus, through a DG a teacher can build a *document* from the resources she interacted with.

An example of a *document* is presented in Gueudet & Trouche (2009, p. 205). In this example, the class of situations faced by a mathematics teacher is to "propose homework on the addition of positive and negative numbers". After looking at several resources such as textbooks and a list of exercises that she has used before, the teacher creates a new list of exercises to use in her lesson. The teacher could modify this list of exercises after seeing how it works in her classroom, and she could reuse it in a new group of students or even in the next school year. After looking at this example, it could be interpreted that the *document* created by the

[42] It can be interpreted as an analogy since during the instrumental genesis a person interacts with and appropriates a tool, whereas the tool shapes the way of thinking and acting of the person. Similarly, during the documentational genesis a teacher interacts with and appropriates a set of resources, whereas the set of resources influences teacher's way of thinking and acting.

teacher is reduced to the list of mathematical exercises that she produced. However, a *document* is not necessarily a physical entity.

A *document* is a scheme (also called *scheme of utilization*) associated with a specific set of resources (in the example above, the *resources* are the textbooks and the list of exercises that the teacher consulted) that guides and determines teacher's action in a given class of situations (in the example the *class of situations* is to propose homework on the addition of positive and negative numbers), across different contexts (*contexts* like the group where she applied the list of exercises and the possible future groups or courses where she could reuse the list). In the example previously mentioned, the creation of the list of mathematical exercises is only a visible part of the document that the teacher has established. There are other non-visible elements that guided and determined the selection and design of the exercises that the teacher included in her list. Such non-visible elements are beliefs and implicit values that drive and lead her actions. Gueudet & Trouche (2009, p. 205) mention an example of these non-visible elements: the idea that "the additions proposed must include the cases of mixed positive and negative numbers, and of only negative numbers".

Thus, a document is associated with a specific set of resources and consists of a visible and tangible part called *usages*, and a non-visible and implicit part called *operational invariants* (Vergnaud, 1998). A document can then be expressed by the following formula:

$$\text{Document} = \text{Resources} + \underbrace{\text{Usages} + \text{Operational Invariants}}_{\text{SCHEME OF UTILIZATION}}$$

A graphical representation of a documentational genesis is shown in figure 18.

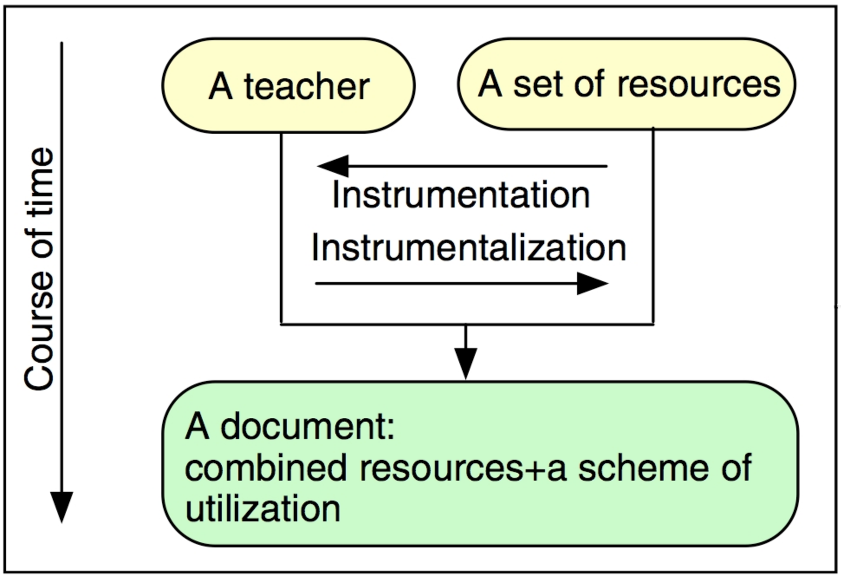

Figure 18. Schematic representation of a documentational genesis. Taken from Gueudet & Trouche (2009, p. 206).

7.1.2 On the concept of documentational orchestration

As I mentioned in the introduction, the concept of *documentational orchestration* that I have developed is inspired by the concept of *instrumental orchestration*. I will start this section referring to the latter concept.

In the paper Trouche (2005a), it is claimed that the schemes of utilization have a social dimension. In this paper Luc Trouche cites the work of Rabardel & Samurçay (2001), where it is affirmed that such schemes are developed and shared in communities, and that may be even the result of explicit training processes. It is then necessary that such "explicit training processes" could be carefully designed to encourage the establishment or modification of schemes of utilization. It is here where the concept of instrumental orchestration appears.

The concept of *instrumental orchestration* (Trouche 2004, 2005a; 2009) arises from the recognition of the need to organise the artefacts available within a given environment, with the purpose of assisting the instrumental genesis of individuals. An instrumental orchestration is defined by two elements (Trouche 2005a, p. 211):

- A set of configurations (i.e. specific arrangements of the artifactual environment, one for each stage of the mathematical situation)
- A set of exploitation modes for each configuration

Now that the concepts of documentation work, documentational genesis, and instrumental orchestration have been introduced, it is possible to provide a more precise characterisation of the concept of documentational orchestration.

Let us first move to the context of mathematics teacher education institutions (like the one where this research was developed). This is a context in which the interaction between mathematics teachers and resources is not spontaneous. In this sort of educational settings it is necessary to organise the resources with which teachers interact, and which are aimed at developing specific aspects of their professional knowledge. Here is where I find important and relevant to use the concept of documentational orchestration.

A *documentational orchestration* (DO) can be defined as the selection and arrangement of resources that a teacher educator (or a group of teacher educators) carry out with the intention of facilitating teachers' documentation work. Such documentation work is aimed at contributing to the development of teachers' professional knowledge. In principle, the structure of a DO should include the two elements that define an instrumental orchestration, namely, configurations and exploitation modes.

Through its configuration, the structure of an orchestration is specified and ordered. By clearly identifying what are the components and stages of an orchestration, it is easier to identify the particular elements of an orchestration that influence and shape teachers' way of thinking and acting. It could be argued that it is possible to explicitly structure and orderly arrange the set of resources within a particular course design without using the concept of DO, however, this is not a concept that should be considered in isolation. The theoretical strength of the concept lies in its connection with the concept of documentational genesis.

The instrumentalization and instrumentation processes are used to understand the way in which a particular orchestration is utilised and appropriated by the teachers, but they also help to understand the kind of effects that the orchestration produces on teachers.

Thus, the documentational orchestration plays a dual role in this research. On the one hand, it is a tool for the *design* of an online course, in the sense that it helps to order and to make explicit the location and functions of the resources that constitute an online course (this aspect was discussed in the chapter 6). On the other hand, the orchestration guides the *analysis* of the operation of an online course, in the sense that forces the researcher to observe the processes of instrumentalization and instrumentation from a micro perspective. That is, it helps to make a kind of zoom-in on the documentational genesis that focuses only on the relationship between such process and the components of an online course. This aspect of the documentational orchestration will be discussed in the section 7.5. Before that it will be illustrated how the processes of instrumentalization and instrumentation are identified within the empirical data.

7.2 Method for analysing the data

As already mentioned in chapter 5, most of the empirical data used in this research are online discussions held by mathematics teachers in asynchronous discussion forums. The analysis of the data generated during the application of the second online course was focused on identifying the emergence of instrumentalization and instrumentation processes. I assumed that the identification of these processes would provide me with information about the operation of the orchestration and its influence on teachers. I particularly assumed that:

- The identification of the *instrumentation* processes would help me to clarify whether or not the emergence of reflections was one of the effects produced by the orchestration. If that were the case, it would be necessary to locate both ends of the arrow representing this process (see figure 18). Using that figure, the end of the arrow would represent a *reflection* that the resources produced in the teachers, whereas the origin of the arrow would represent the particular resource or resources that produced such reflection.
- The identification of the *instrumentalization* processes would help me to understand how teachers use and relate to the resources of the orchestration. This would provide me with relevant information as a designer, since I could verify which resources were used as intended and which resources were not.

These were my expectations prior to the application of the theoretical constructs of instrumentalization and instrumentation processes. Now I would like to clarify the method that I followed to identify these two processes within the data:

Firstly, it is necessary to transit through an acquaintance stage just as the one described in chapter 5 (see section 5.2.2). The aim of this stage is to become familiar with the contents and development of each discussion forum.

Secondly, based on the configuration of the orchestration (see figure 16, chapter 6), the identification of the possible manifestations of instrumentalization and instrumentation processes in the data is undertaken. The reader should recall that in the configuration, the non-human elements (hereinafter interpreted as *resources*) and the purposes of each stage of the orchestration were specified. Drawing on such stages, the following two questions are answered:

1. Were the resources of the stages utilised as expected?
2. Did the stages produce the expected "effects" on teachers?

To illustrate these two points, let me take as an example the third stage of the orchestration described in the section 6.2.3 of the sixth chapter. At this stage, teachers should find a general factorisation for the expression $x^n - 1$ using both, instrumented and the paper-and-pencil techniques. Both techniques were the non-human elements central to this stage.

Regarding question (1), it was expected that the teachers using paper-and-pencil techniques would perform a polynomial division and find that the expression $x^n - 1$ can be factored as $(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)$. Moreover, teachers using the command Factor technique were expected to discover that the factorisation of $x^n - 1$ sometimes produces more than two factors. It was also expected that, in an effort to identify the necessary conditions for this to happen, teachers would complement the Factor technique with paper-and-pencil techniques, such as polynomial division. Thus, if teachers interact with the resources differently than expected by the designer, or beyond what they were instructed; such actions would be regarded as *instrumentalization* processes. In the same way, if teachers appropriate and/or modify the resources in ways not anticipated by the designer, or adding new resources not included in the initial configuration of the orchestration, such actions would be regarded as *instrumentalization* processes.

With respect to question (2), it was anticipated that by means of the third stage teachers would compare and discuss the advantages and disadvantages of both techniques. Particularly teachers were expected to highlight the *pragmatic value* of the instrumented techniques (speed and efficiency for performing calculations), but it was also assumed that some of them would recognise some kind of *epistemic value* in the instrumented techniques (such as the possibility of promoting a more experimental and inquiry oriented approach to the study of algebra). Thus, if teachers somehow manifested any of these "expected effects", then one could speak of a *instrumentation* process. However, as will be shown below, in this category it was also included those "unintended but desirable effects" that the resources of the orchestration produced.

In the following two sections, the instances of the instrumentalization and instrumentation processes that were identified through this method will be presented. After that, a discussion of the implications of those findings is presented.

7.3 Instances of instrumentalization processes

As mentioned previously, an instrumentalization process refers to the moment when a teacher appropriates and/or modifies the set of resources with which he or she is interacting. It also refers to those situations where a teacher adds new resources not included in the original orchestration.

During the analysis of the interactions between teachers and the resources of the orchestration, some instances of instrumentalization processes were located. Most of them were cases of appropriation of the discussion forums that the teachers developed in order to communicate mathematical ideas. When I use the term "appropriation of the discussion forums" I mean that the teachers begin to get familiar with and handle with ease the tools offered by the discussion forums.

For instance, the writing tools that the discussion forums offer are limited with regard to the expression of mathematical symbols. Users only have at their disposal tools for writing subscripts and superscripts, and a small collection of special characters, among which some mathematical symbols are included (see figure 19).

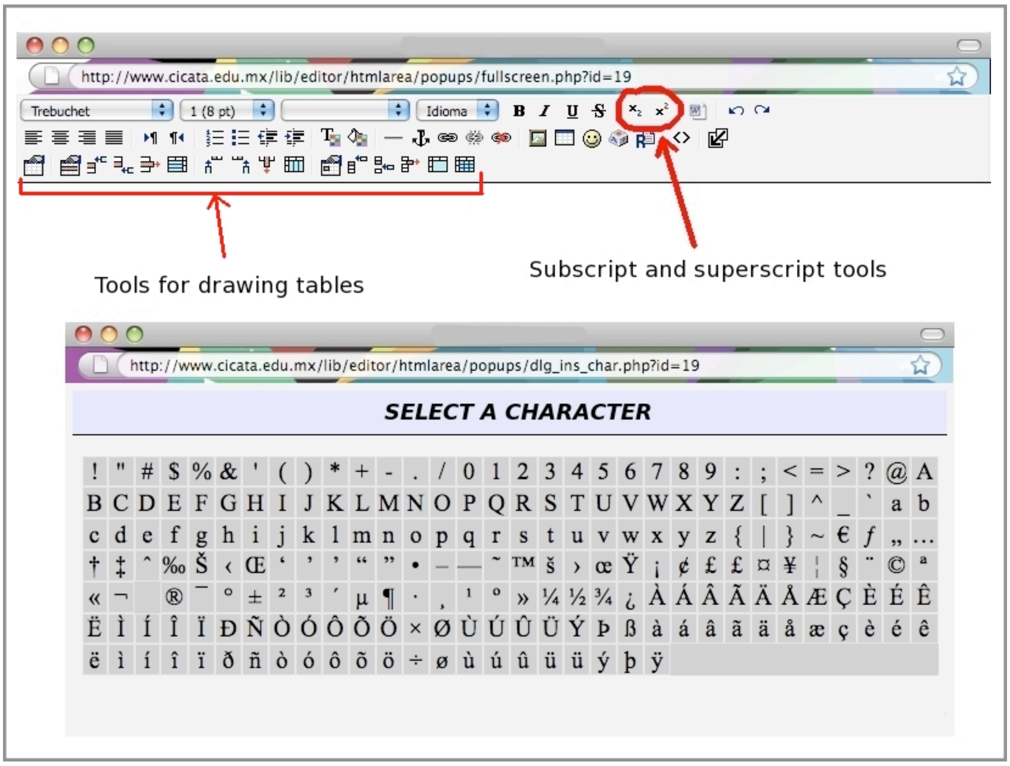

Figure 19. Writing tools that are accessible in the discussion forums. The available tools for representing mathematical symbols are limited.

My interpretation is that the limitations of such writing tools was one of the factors that drove teachers to seek creative ways of communicating mathematical ideas within the discussion forums. This led to instances of instrumentalization processes such as those presented below.

7.3.1 Case 1: Adapting the communication tools of the forum

During the third stage of the orchestration, a teacher who was using paper-and-pencil techniques to find a general factorisation of $x^n - 1$, shared her findings with her colleagues. In the forum she said that the expression $x^n - 1$ is always the product of $x-1$ by a polynomial of degree $n-1$. But she also stated that when n is an even number, it is possible to further factorise the expression. That is, it is possible to obtain more than two factors (except for $n = 2$). This however is not reflected in the mathematical expressions that she used to complement her statements (see figure 20).

if n=1;	$(x^1-1) = (x-1)$
if n=2;	$(x^2-1) = (x-1) . (x +1)$
if n=3;	$(x^3-1) = (x-1) . (x^2 + x + 1)$
if n= 4;	$(x^4-1) = (x-1) . (x^3 + x^2 + x + 1) =$
if n=5;	$(x^5-1) = (x-1) . (x^4 + x^3 + x^2 + x + 1)$
......	.....
if n e N;	$(x^n-1) = (x-1) . (x^{n-1} + x^{n-2} + x^{n-3} +.... + x + 1)$

Figure 20. Mathematical expressions developed with the writing tools available in the forum. It is notable the use of superscripts to denote exponents, and the use of the letter "e" as a substitute for the symbol "$\in$".

The teacher only showed her conclusions, but she did not explain the technique she used to reach them. This situation caused that one of the participants in the forum asked: Why in the case of an even exponent it is always possible to obtain another factorisation? Thus, the teacher was required to clarify the method by which her conclusions were reached. She answered to the question in this manner:

[27]

Theme: Re: Team 2. "Paper and pencil technique"
From: Susana
Date: Thursday, 27th of November 2008, 05:51

Let me see if I can explain myself through these examples. If you do not understand I will try again. For example:

If n = 4; $(x^4 - 1) = (x - 1) . (x^3 + x^2 + x + 1)$

The polynomial $(x^3 + x^2 + x + 1)$ is divisible by $(x + 1)$, and dividing it or by applying Ruffini's rule, you get:

	1	1	1	1
-1		-1	0	-1
	1	0	1	0

Then the resulting factorization is:

If n = 4; $(x^4 - 1) = (x - 1) . (x^3 + x^2 + x + 1) = (x - 1) . (x + 1) . (x^2 + 1)$

[…]

The teacher showed several examples like the above to clarify her arguments. What I want to emphasise here is that, in order to illustrate Ruffini's rule[43], the teacher did not resort to external communication tools that facilitate the expression of mathematical ideas and techniques (such as an equation editor from a text processor), as teachers usually do when communicating this sort of mathematical ideas in the forums. Instead, the teacher chose to make the most of the communication tools available in the forum. In this case, besides using superscripts to denote exponents, she utilised the tools for drawing tables in a creative way (see figure 19). Particularly, the teacher inserted a table in her utterance (see [27]) but

[43] Ruffini's rule is an algorithm which allows you to quickly perform the division of a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ by the binomial $Q(x) = x - r$.

leaving some cells with no edges to illustrate the application of Ruffini's rule.

This situation is considered as an instance of an *instrumentalization* process. This because it manifest that the teacher have reached a level of ownership and familiarity with the communication tools of the forum that only few teachers have reached. Such appropriation of the resources allows the teacher to communicate her mathematical ideas in a more clear and efficient manner.

However, an *instrumentalization* process not only refers to situations in which teachers become familiar with the resources available within an orchestration. It also refers to situations in which teachers incorporate new resources to the orchestration. This is illustrated through the next case presented.

7.3.2 Case 2: Using a YouTube video

Another case of an instrumentalization process occurred when a teacher who used paper and pencil techniques was looking for a general factorisation of $x^n - 1$. The teacher explained in the discussion forum that she had been applying Ruffini's rule to try to find a general factorisation. Then one of her colleagues asked her: Can you tell me in what book I could find Ruffini's rule? The teacher responded to the question as follows:

> [28]
> *Theme:* Re: Team 2. "Paper and pencil technique"
> *From:* Norma
> *Date:* Wednesday, 26th of November 2008, 00:09
>
> Nice to meet you Homero, how are you?
>
> You may already know Ruffini's rule (as we call it here [in Argentina]) but with a different name. It is a shortened way of solving [polynomial] divisions

having the form P=(x)/(x+-b) […] To be consistent with this course, I will not recommend you any book, I will give you a direct link to a youtube video.

A picture is worth a 1000 words, don't you think? ☺

http://es.youtube.com/watch?v=RViiUlWty8M

Norma

This situation is interpreted as an example of an instrumentalization process, because the teacher introduces a novel resource into the orchestration, namely, a link to a video hosted on YouTube. As the reader can confirm by following the link included in the utterance [28], the video shows a person who is explaining (in Spanish) and illustrating step by step how to apply Ruffini's rule for a particular polynomial. The teacher Norma uses this video as a means to communicate to her colleague the mathematical technique that she has been applying in the factorisation process. Even though the teachers educators from the CICATA program had previously used this website to post video messages, this was the very first time that we saw a teacher using this sort of videos as a means for communicating mathematical ideas.

In the section 7.5 the implications of the instrumentalization processes that have been presented in this section will be discussed. Before that, a couple of cases where instrumentation processes were identified are presented.

7.4 Instances of instrumentation processes

In general, an *instrumentation* process occurs when the resources with which a teacher interacts shape and influence her professional activity and knowledge. In the particular case of this research, the focus is on identifying instrumentation processes in which the reflections of a mathematics teacher, are influenced by the resources with which he or she interacts along the online course.

During the data analysis, it was found that the instrumentation processes are not as frequent as the instrumentalization processes. In this section. In this section the only two cases of instrumentation processes that were detected during the data analysis are presented. The first case shows an instrumentation process which was triggered by an instrumentalization process.

7.4.1 Case 3: Triggered by an instrumentalization process

Two teachers named Marta and Rosa are in the third stage of the orchestration and they start the search for a general factorisation of $x^n - 1$, but applying the command Factor technique. In the discussion forum, Marta suggests to start the inquiry by factoring particular cases of $x^n - 1$. Rosa supports the suggestion and decides to divide the factorisations into two cases: the case when n is even and the case when n is odd. Later, Rosa posts a message in the discussion forum in which, through a text file attached to the message, she reports to her colleague the exploration that she has conducted by using the command Factor of the software. The results obtained by Rosa when applying the command Factor are concentrated in table 6:

For n even
$x^2-1=(x-1)(x+1)$
$x^4-1=(x-1)(x+1)(x^2+1)$
$x^6-1=(x-1)(x+1)(x^2-x+1)(x^2+x+1)$
$x^8-1=(x-1)(x+1)(x^2+1)(x^4+1)$
For n odd
$x^3-1=(x-1)(x^2+x+1)$
$x^5-1=(x-1)(x^4+x^3+x^2+x+1)$
$x^7-1=(x-1)(x^6+x^5+x^4+x^3+x^2+x+1)$
$x^9-1=(x-1)(x^2+x+1)(x^6+x^3+1)$

Table 6. Factorizations of x^n-1 obtained by applying the command Factor. It is difficult to identify a pattern since the number of factors obtained is not regular.

Just after presenting the results included in the table 6, Rosa writes the question: "Do you see a pattern?", and then she continues commenting:

> [29]
>
> [...] I'm going to shift your attention a bit. It came to my mind to try to visualise what happens to the graphical behaviour. [...] A special case is when n is equal to 1 [here the teacher inserts in her text the graph of $y=x-1$].
>
> With n even [the teacher inserts figure 21]. With n odd [the teacher inserts figure 22]. I do not know to what extent is possible to visualise it, but I share with you what I have found. I am still working on it.

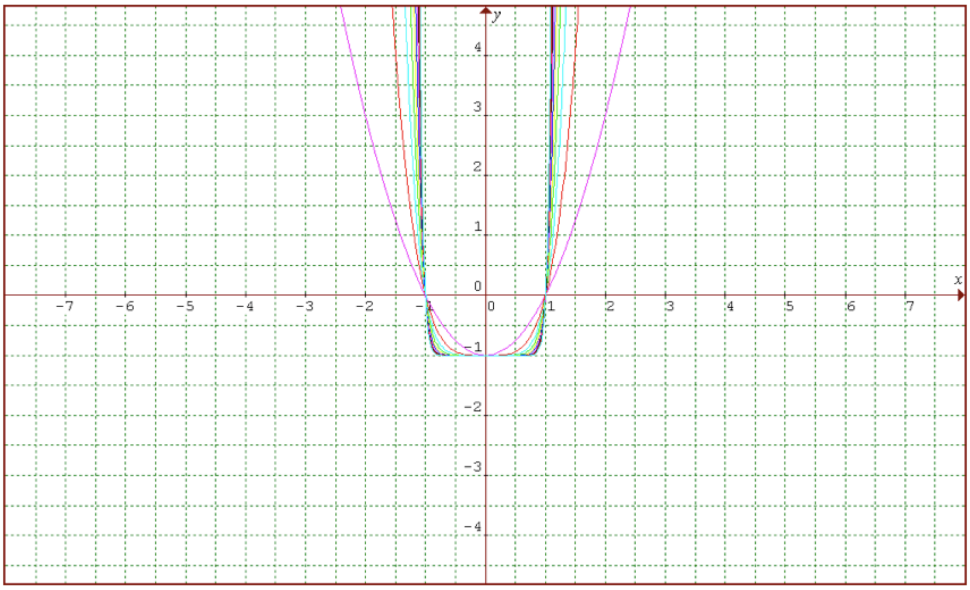

Figure 21. Graph of the family of curves defined by $y = x^n - 1$ for some even values of n.

Figure 22. Graph of the family of curves defined by $y = x^n - 1$ for some odd values of n.

After reading this message, Marta responds to Rosa with another comment in the discussion forum:

> [30]
>
> *Theme:* Re: Team 1. "Command Factor technique"
> *From:* Marta
> *Date:* Tuesday, 25th of November 2008, 23:58
>
> Nice to meet you Homero, how are you?
>
> Hi Rosa: I took the opportunity to write you from my workplace. I think that what you did with respect to discriminate if the exponent is even or odd is right. With respect to the graphs, which I find very interesting, I do not know how to integrate them since they are just asking us to factorise. But we will see. [...]

This situation is considered a case of an instrumentalization process, because teacher Rosa introduced the use of Cartesian graphs for the solution of the activity (figures 21 and 22). Rosa did not limit herself to the use of the command Factor as requested in the guidelines of the activity. In fact the graphs 21 and 22 were produced by using a different software than the officially used in the course.

It seems that Rosa decided to explore the graphical context because the factorisations provided by the software (table 6) did not allow her to glimpse a general factorisation. The question "Do you see a pattern?" directed to her colleague confirm this observation. It also seems that Rosa failed to establish a link between the factorisations obtained and the graphs that she produced. Her comment in [29] gives that impression.

Apparently Marta does not find a clear association between the graphs and the factoring task either. In [30] she commented: "I do not know how to integrate them since they are just asking us to factorise. But we will see". In fact, Marta and Rosa did not recur in their subsequent explorations to the use of such graphs. They tried to produce a general factorisation from the particular results obtained through the application of the command Factor. After posing and verifying several hypotheses, they concluded that for both, odd and even values of n:

$$x^n - 1 = (x-1)\left(x^{n-1} + x^{n-2} + \cdots + x + 1\right)$$

Upon concluding their exploration, Marta and Rosa edited a report that was shared with the sub-team that worked on the same factorisation activity, but utilising paper-and-pencil techniques. This report presented the above-mentioned conclusion along with the performed explorations, including the graphs shown in figures 21 and 22. In turn, the sub-team that worked with the paper-and-pencil techniques found two different factorisations. For n odd the teachers found that:

$$x^n - 1 = (x-1)\left(x^{n-1} + x^{n-2} + \cdots + x + 1\right)$$

And for n even they concluded that:

$$x^n - 1 = (x-1)(x+1)\left(x^{n-2} + x^{n-4} + \cdots + x^2 + 1\right)$$

After receiving Marta and Rosa's report, a member of the sub-team working with paper-and-pencil expressed the following in the forum:

[31]

Theme: Re: Area for the general discussion
From: Federico
Date: Saturday, 29th of November 2008, 22:59

[…] I am trying to integrate all the things done by the two sub-teams. I have two comments:

The conclusion that in the even cases the factorisation is the same than in the odd cases, although strictly correct, I think that we must also express that

> there is a common factor to all the even cases, it is x+1 and it could be expressed as:
>
> $x^n-1=(x-1)(x+1)(x^{n-2}+x^{n-4}+...+x^2+1)$ when n is even
>
> The second observation is that your graphs are amazing because they confirm the previous observation and help to understand that the only real roots are -1 and 1 generating the factors x+1 and x-1, the first one only for even cases. [...]

The utterance [31] shows that the graphs that Rosa produced, although relatively ignored in her own sub-team, acted as a new resource that allowed teacher Federico to assign a new meaning to the factorisations that he found during his own explorations with paper-and-pencil. This mathematical clarification experienced by Federico is considered as an *instrumentation* process, but more particularly a *mathematical reflection*. It is an instrumentation process since the graphs introduced by Rosa (here considered as a new resource) influenced the mathematical insight experienced by Federico. This case shows that an instrumentalization process (the introduction of the graphs) has the potential for creating new resources. And in addition, such new resources may influence the way of thinking of the teachers who interact with the new resources.

7.4.2 Case 4: Theoretical ideas from mathematics education

After analysing teachers' asynchronous discussions produced on the stages three and four of the orchestration, it became clear that in many of them only the *pragmatic value* of instrumented techniques was being emphasised. In other words, teachers perceived the software as a tool that facilitates the execution and verification of algorithms, but not as a tool that can serve as a means for mathematical inquiry and the construction of mathematical knowledge. Such positions can be illustrated by the comments that some of the teachers expressed at different times of the course. For example, during the composition of the sub-teams that would carry out the factorisation task proposed in stage three, one of the teachers decided to join another teacher who would like to address the task by using paper-and-pencil. He expressed his interest through the following comment:

> [32]
>
> *Theme:* Re: Area for the general discussion
> *From:* Francisco
> *Date:* Monday, 24th of November 2008, 19:35
>
> Hi colleague, even though I support the use of calculators I am convinced that the proper use of calculators previously requires to have understood how the things are done. Also I would like to team up with you. If you do not mind we could make a team […]

An interpretation of the phrase "I am convinced that the proper use of calculators previously requires to have understood how the things are done" is that this teacher perceives technology (or in this case calculators) as an element that should be used in the classroom after the work with paper-and-pencil. The phrase suggests that this teacher did not perceive instrumented techniques as a means of producing knowledge. This idea or position is interpreted here as an operational invariant that this teacher associates with the use of technology in the teaching of mathematics.

The previous position was reiterated by the same teacher in the fourth stage of the orchestration. At this stage the working groups had to discuss the potential impact that technology would have on Juan Castro's lesson plan (see the description of the fourth stage on the orchestration in the section 6.2.4 of the sixth chapter):

[33]
Theme: Re: Tasks and techniques
From: Francisco
Date: Friday, 5th of December 2008, 19:03

Hello colleagues:

I agree with Rosa on the usefulness of the calculator (ClassPad) regarding that it saves a lot of work [...] In general, when there is discussion of this topic I always conclude that it is important for students to first learn the methods by hand, let us say pencil and paper. [...]

For the teacher educators who were observing the evolution of the course, it was clear that after the teachers had gone through the initial stages of orchestration, most of them only highlighted the *pragmatic value* of instrumented techniques without mentioning its possible *epistemic value*. This situation was explicitly addressed during a meeting that the teacher educators held three days after the start of the fourth stage of the orchestration. At this meeting we agreed that, during the fifth stage of the orchestration in which the work of Lagrange (2005) would be discussed along with the mathematics teachers, it should be explicitly addressed the concepts of pragmatic and epistemic value. This is the way in which such discussion was initiated by one of the teacher educators:

[34]
Theme: Technology in the mathematics classroom?
From: Jaime
Date: Wednesday, 10th of December 2008, 01:06

Hello colleagues:

Today we will start a dialogue on the use of technology taking as a point of departure two main issues. Your experiences with the course activities and the reading of the article by J. B. Lagrange.

We will try to reflect upon two main concepts included in the paper. What he calls epistemic and pragmatic values.

> You have confronted situations using paper and pencil and using a calculator. What is obtained from one setting and from the other one [?] [...]

Francisco, the teacher who wrote the comments shown in [32] and [33], had the following reflection in response to Jaime's comment in [34]:

> [35]
>
> *Theme:* Re: Technology in the mathematics classroom?
> *From:* Francisco
> *Date:* Saturday, 13th of December 2008, 04:16
>
> Hello colleagues:
>
> Until I read Lagrange's article I only applied it [the technology], using the terminology of the article, in a pragmatic way. I even felt that without a prior knowledge the use of tools such as CAS and/or calculators did not help to generate learning, i.e., I did support the use of these tools but apparently only attaching value to their pragmatic aspect. In integral calculus I encouraged the use of these tools in all the required calculations up to derivation. In differential equations I incentivise its application in the calculation of integrals and so on. So I was very surprised that the article emphasises the epistemic aspect of these applications. Partly he was right, because the epistemic application apparently requires planning and construction of new specific activities that do not arise naturally from the teaching with paper and pencil. I would like to conclude this contribution leaving the reflection and concern of how a methodology for applying the epistemic value should be.
>
> Best wishes
>
> Francisco

Francisco's comment number [35] suggests that the concepts of epistemic and pragmatic values helped him to identify or to be aware of the existence of some of his values associated with the use of technology in mathematics teaching. If we compare his comment issued in [35] with those expressed in [32] and [33] it becomes clear that this teacher has identified the pragmatic approach that he has taken toward the use of technological tools.

This was not the only moment in which Francisco reflected on his own position regarding the use of technology. Some hours after the publication of his comment [35], he referred to the intervention of one of his colleagues, where she describes in general terms the way she has been using mathematical software with her students. Francisco said:

> [36]
> *Theme:* Re: Technology in the mathematics classroom?
> *From:* Francisco
> *Date:* Saturday, 13th of December 2008, 17:21
>
> […] [A]pparently Mariana has already considered a series of tasks or activities to learn how to solve systems of linear equations through matlab. Taking this example it seems that to assign a pragmatic value is: once you know how to solve these systems, you use the tool to facilitate the algorithms. To give an epistemic value means to learn the involved concepts through the use of the tool. I had not visualised applying the technology in this sense; I have only promoted its pragmatic value. What motivates me now is learning to develop tasks and learning methods, I have the impression that what Mariana is considering, points in that direction. [...]

The utterance [36] also suggests that the teacher has identified his own pragmatic approach towards the use of technology. He even shows a willingness to explore the possible epistemic value of instrumented techniques. This situation is interpreted as a *didactical reflection*. That is, the teacher has explicitly considered his own teaching practice in light of the concepts of pragmatic and epistemic values. In addition, the teacher has made an association of his own teaching behaviour with one of these values. This situation is also interpreted as a potential change in the operational invariants that this teacher associates with the use of technology, in the sense that he seems willing to try out other ways of using technology in his teaching practice. Such potential change seems to have been motivated by some of the resources with which he interacted, particularly by the concepts of epistemic and pragmatic values of a technique presented in the article by Lagrange (2005). This apparent change in teacher's perception of the use of technology in mathematics teaching is considered an *instrumentation* process.

7.5 A discussion on the documentational approach

Before discussing the implications of the instrumentalization and instrumentation processes previously presented, it is necessary to briefly discuss how the instrumental approach has been applied in this research.

At the beginning of this chapter I commented that, in order to analyse the interactions between mathematics teachers and the non-human elements of an online course, some of the theoretical concepts provided by the documentational approach were applied. Let me now elaborate on this point.

In their exposition of the documentational approach, Gueudet & Trouche (2009) pay particular attention to the kind of document that a teacher creates when interacting with a particular set of resources, and how this document evolves over time. They emphasise that the documentational genesis is a process that results in a dialectical relationship between documents and resources:

> "A documentational genesis must not be considered as a transformation with a set of resources as input, and a document as output. It is an ongoing process. […] [A] document developed from a set of resources provides new resources, which can be involved in a new set of resources, which will lead to a new document etc. Because of this process, we speak of the *dialectical* relationship between resources and documents" (p. 209, emphasis in the original).

Gueudet & Trouche (2009) represent graphically this process as a helix:

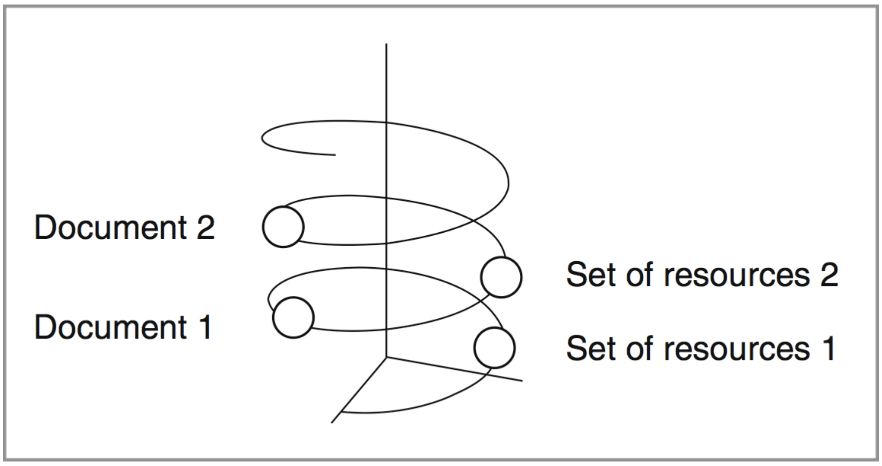

Figure 23. This helix represents the resource/document dialectical relationship and its evolutionary nature. The illustration is taken from Gueudet & Trouche, 2009, p. 206.

What I am emphasising here is that Gueudet & Trouche (2009) focus on the long-term evolution of the relationship resource/document. In terms of figure 22, this means to focus at once on several rounds of the helix.

Kieran (2009) has suggested that it may also be important to focus on local points of the helix shown in Figure 22:

> "Notwithstanding Gueudet and Trouche's insistence that documentational genesis is an ongoing dialectical process and not a transformation, I still wondered as I was reading the paper whether or not – at a more atomic level – documentational genesis could not be viewed as a set of transformations, albeit interrelated. This would permit the study of the impact of the introduction of salient new resources on the existing documentational systems of the teacher at the very 'moments' that the resources are being introduced. In other words, in addition to focusing at once on several rounds of the evolutionary helix – as do Gueudet and Trouche, I am wondering whether a fine-grained zoom-in on a much smaller part of the helix, say a point or a short arc, might also contribute to

> our knowledge of teachers' development of documents. Depending on where and when within the helix these zoom-ins are carried out, the descriptions could capture in a detailed manner the instrumentalization or instrumentation dimensions of the process of documentational genesis, that is, the ways in which the teacher is shaping or being shaped by these resources." (Kieran, 2009, p. 2)

I completely agree with this observation. In fact, this was the way the documentational approach was used in this research. Mi statement about applying "some theoretical concepts provided by the documentational approach" means that I did not focus neither in the identification of the documents created by the teachers, nor in monitoring its evolution over time. As the reader could verify through this and the previous chapter, my attention was focused on: firstly, to select and arrange a particular set of resources; and secondly, to locate the instrumentalization and instrumentation processes generated from that arrangement of resources (the zoom-ins that Kieran, 2009, suggests).

It was necessary to focus on the location of these processes to try to shed light on how teachers interact with a set of resources, and how such interactions may influence the emergence of reflections. Although it is important to determine the type of documents that teachers create from a particular orchestration, it would not have been possible to accomplish such task within this research. As discussed in section 7.1.1, a document has a visible and tangible component called usages. This component can only be determined through the observation of teacher's actions in the classroom (Gueudet & Trouche, 2009, p. 209). Thus, to identify a document that may be produced from a particular orchestration it would be necessary to use a different research method. However, this should not be perceived as a limitation of the study. As discussed below (and as suggested by Kieran, 2009), making zoom-ins to the relationship resource/

document provides relevant information on the orchestration and its relationship with the teachers who use it.

7.6 Implications of the findings

Focusing on identifying the instrumentalization and instrumentation processes associated with a particular orchestration allowed me to obtain relevant information about the type of resources that have the potential to influence the emergence of mathematics teachers' reflections. Besides, this type of analysis also provided me with information regarding the operation of the orchestration, regarding developments in the management of computational tools by some teachers, and even information about the nature of online collective documentation work. In the following each of these aspects is discussed.

7.6.1 On the nature of the online collective documentation work

As described in section 7.1.1, the term collective documentation work (CDW) refers to the type of collective work where teachers interact with and share a common set of resources. During the implementation of the orchestration, teachers developed a very particular type of CDW: one that is based on the use of the Internet. Therefore is referred as online collective documentation work. By identifying the instrumentalization and instrumentation processes it was possible to discover some aspects of the nature of the online CDW. One of the most important aspects is the fact that, in the online CDW not only the resources acquire a public and collective character. Also the teachers’ actions on these resources become tangible, public and shared. An example of this is the case three presented in section 7.4.1. This case illustrates how an instrumentalization process (in this case the incorporation of some graphical representations in the discussion about the general factorisation of $x^n - 1$), although produced by a single person, has the potential to affect the activity of other teachers.

What I want to emphasise here is that the Internet favours that teachers' actions and ideas acquire a tangible nature. It is a kind of reification. This quality favours that these actions and ideas become new resources of the orchestration. In other words, this is an environment where the production of resources cannot be monopolised by the designer(s) of the orchestration. This situation, as will be discussed next, has implications for the emergence of reflection in an online setting.

7.6.2 Resources that influenced teachers' reflections

Locating the type of non-human elements that have the potential to influence the emergence of teachers' reflections was one of the concerns that drove this second phase of the research. As expected, the identification of the instrumentation processes provided information on the matter. Unfortunately, only two instances of instrumentation processes were found (cases 3 and 4), but I claim that these two cases enhance our understanding of the relationship between non-human elements of an online course and the emergence of mathematics teachers' reflections.

The third case presented in section 7.4.1 shows a situation in which a teacher's idea or insight (in this case to look for visual support in trying to understand the algebraic behaviour of the factorisation of $x^n - 1$) was materialised and shared within an online setting. It was in that moment that teacher's idea became a new resource of the orchestration, capable of nourishing the documentational work of her colleagues. It even produced a mathematical reflection in one of them (see [31]).

In the second case shown in section 7.3.2, another example of an incorporation of a new resource was presented. It is likely that the

YouTube video introduced by teacher Norma (see [28]) had produced some kind of reflection or mathematical understanding in Homero regarding the operation of Ruffini's rule. Unfortunately I have no such evidence[44]. However, both cases are useful to argue the following conclusion: In an online setting, teachers' reflections can be detonated by the resources introduced by other teachers. In addition, the orchestration designer cannot control the incorporation of such resources. In other words, the emergence of teachers' reflections within an online setting is a complex process, sensitive to different inputs, which cannot be completely controlled.

The above mentioned conclusion may sound disheartening to a teacher educator who seeks to understand how reflections can be encouraged. However as I will argue next, there are other kind of resources, easier to control, which have the potential to trigger reflections in mathematics teachers.

In the section 7.4.2, the case of a teacher named Francisco who experienced a didactical reflection was shown. This reflection allowed him to make visible some of the values that he associated with the use of technology in mathematics teaching, and the influence of such values in his own practice. The most important point here is that this reflection was triggered by the encounter that this teacher had with the concepts of pragmatic and epistemic values discussed in Lagrange (2005). Here I am not only claiming that these concepts have the potential to promote didactical reflections on the use of technology, my claim is even more general: I think that there are other concepts from mathematics education

[44] Homero made no comments in the forum on the content of the video.

research which can promote the development of mathematics teachers' reflections.

Let me elaborate on this last statement. In the third chapter of this dissertation, a review on the concept of reflection was presented. There, some of the conditions that are claimed to promote the emergence of reflections (according to previous research results) were mentioned (see section 3.3.4). Particularly relevant are the statements of Mewborn (1999) and Hodgen (2003), who refer to the ability of being *distanced* or *decentred* from our own practice or actions as a condition for the appearance of a reflection. I think that some of the concepts from mathematics education research could provide teachers with such a distance, allowing them to see their own practice from a detached perspective. I will try to justify this last statement referring to the concept of *cultural model* discussed in Presmeg (2007):

According to D'Andrade (1987) a cultural model is "A cognitive schema that is intersubjectively shared by a social group" (p. 112). D'Andrade adds: "One result of intersubjective sharing is that interpretations made about the world on the basis of the folk model[45] are treated as if they were obvious facts of the world" (p. 113). Presmeg (2007) uses this concept to explain why certain beliefs about mathematics have existed within the mathematics community which are invisible or unnoticed. She claims: "The well-known creativity principle of making the familiar strange and the strange familiar […] is necessary for participants [of a social group] to become aware of their implicit cultural beliefs and values, which is why the anthropologist is in a position to identify the beliefs that are invisible to many who are within the culture" (Presmeg, 2007, p. 443). I would

[45] D' Andrade (1987) uses the terms "cultural model" and "folk model" to refer to the same concept.

complement the above quotation stating that the outsider (in this case the anthropologist) is not only able to identify the beliefs and values that are invisible to the people within a particular culture; but also, through a culture comparison (his own culture and the one he is observing), he may be able to identify beliefs and values that are invisible or perceived as "normal" in his own culture! I conclude this from my personal experience as a Mexican living in Denmark for three years, but also from my own experience as a "foreign mathematics educator" coming from an academic culture which is different to the Danish academic culture in many ways. As I was entering into the Danish culture, it was inevitable to make comparisons and produce reflections on the values of my culture and my values as a person.

In a similar way, I think that when mathematics teachers are introduced to the "mathematics education research culture" (through the study of its theoretical concepts, its results, and its products), teachers have the opportunity to distance themselves from their own teaching culture and view it with other set of lenses. A set of lenses provided by the mathematics education research culture. In fact I think that the concepts of epistemic and pragmatic values played such a role during the reflection experienced by the teacher Francisco. The concepts enabled him to stray from his own practice, helping him to identify the type of conception (pragmatic or epistemic) that he had on the use of technology, and even prompted an interest in changing his approach to the use of technology in mathematics teaching.

There are other researchers claiming that the study of concepts and theories from mathematics education research promotes critical reflection on our own beliefs and practices as mathematics educators. Even (1999) shows a study in which teacher's leaders and in-service teacher educators were familiarised with mathematics education research literature (through the reading, presentations and discussions of research articles), as a means to challenge existing conceptions and beliefs about learning and teaching of mathematics. She concludes that in some cases an intellectual restructuring and change in knowledge and beliefs was achieved. In other cases the academic knowledge contributed to an actual change in teaching practice.

In a similar study, Tsamir (2008) reported various experiences with pre-service mathematics teachers who were introduced to the study of mathematics education theories as a means to promote their mathematical knowledge, their pedagogical knowledge and their teaching. In her conclusions Tsamir (2008) reports that one of the teachers involved in the study "used her theory-based knowledge to critically reflect on her own reasoning" (p. 227).

If one accepts that the theoretical concepts from mathematics education research have the potential to encourage the emergence of teachers' reflections, then a question naturally arises: what kind of theoretical concepts must be used for this purpose? Tsamir (2008) raises similar questions, without providing a specific answer. Of course these questions deserve further investigation, however, it is possible to formulate a hypothesis: I believe that the type of theoretical concepts that can help teachers to reflect on their own practice and values, must be concepts that seem applicable to them. In other words, teachers need to find some relationship or application between such concepts and their own teaching practice. Thus, it is likely that theoretical concepts with little or no relation to teachers' practice will not serve for this purpose. But I insist, it is necessary to continue researching this issue.

7.6.3 Informing the redesign of the online course

Another contribution of the study of instrumentation processes is that they can provide relevant information regarding the operation of a particular orchestration. Such information is particularly useful for the refinement and redesign of the orchestration. Let us consider the fourth case discussed in the section 7.4.2 as an example.

Since the information obtained through the instrumentation process suggests that the concepts of pragmatic and epistemic values have the potential to trigger didactical reflections on the use of technology in mathematics teaching, then it seems appropriate to include the explicit discussion of these concepts with teachers, in future versions of the orchestration.

Thus, a documentational orchestration can be regulated and evolve through the feedback obtained after its application. Such feedback is represented by the instrumentalization and instrumentation processes that are manifested during the different stages of the orchestration.

7.6.4 Teachers' development on the use of computational tools

Cases 1 and 2 (see sections 7.3.1 and 7.3.2) show instrumentalization processes that reflect the level of appropriation of computational tools that some teachers have developed. Although it was not the focus of this research, I think that if we look at the instrumentalization processes that a teacher expresses over time (particularly the way the teacher uses and appropriates the available computational tools), this would provide us with information about the development of this teacher in the management of computational tools.

This idea arose when I observed the way in which a teacher communicated graphic elements to his colleagues. For example, when this teacher was doing some graphical explorations related to the activity of the paper airplane problem for the mathematical modelling course (See chapter 4, section 4.2.2), the teacher made a graph using paper and coloured pencils. Later he scanned the piece of paper and attached the resulting file in one of his comments during the asynchronous discussions (see figure 23).

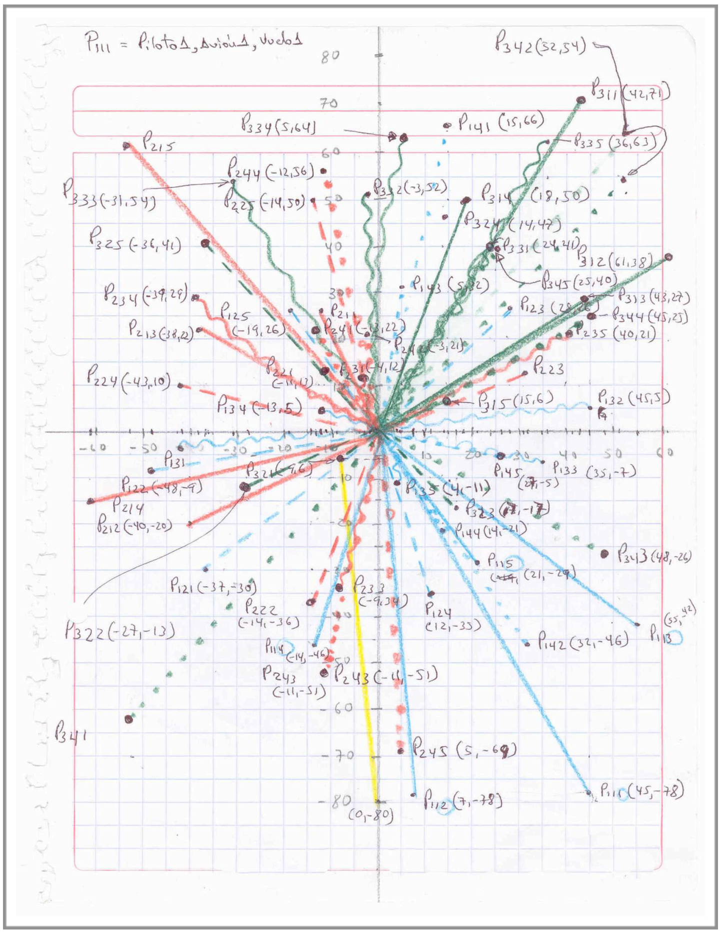

Figure 24. Graphical representation made by one of the teachers participating in an online course. Instead of using any of the computational tools to draw such a graph, the teacher used paper and coloured pencils. My interpretation is that this was due to his lack of experience in using computational tools.

Seven months later, at the beginning of the course on the use of technology that was analysed in this chapter, the same teacher was using other kind of tools to draw his graphs. For instance, the figure 25 shows a graph that the teacher drew to represent two systems of linear functions. This time the teacher used the drawing tools available in a word processor:

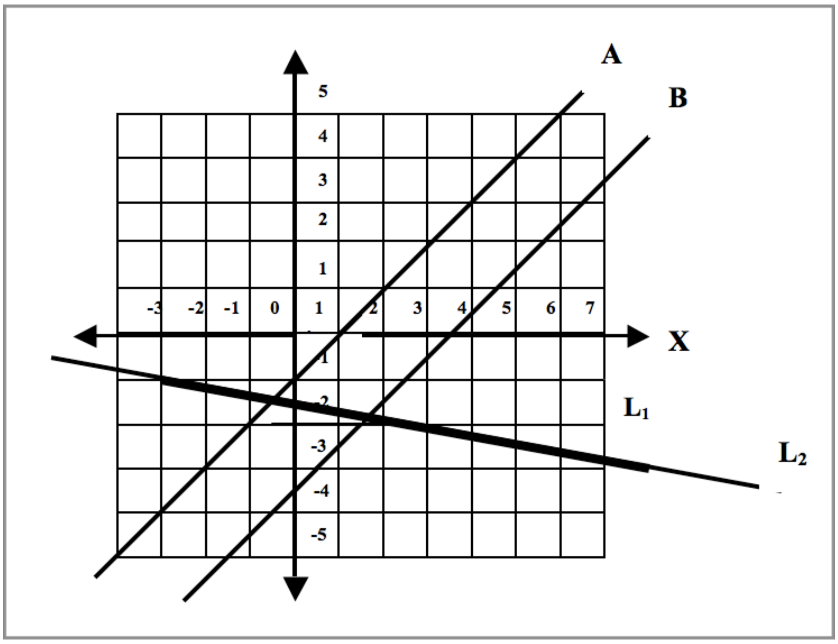

Figure 25. This is a graph made by the same teacher who made the graph shown in figure 24. Here it is possible to perceive an effort to use a different kind of drawing tools.

With this example I want to illustrate how a long-term observation of instrumentalization processes manifested by particular individuals could inform us about their development in the management of technological tools. This is a competence that although is not directly related with their pedagogical or mathematical knowledge, it is important for their general professional development.

7.7 Further development of the approach

The application of the concepts of instrumentalization process and instrumentation process proved to be useful to observe the influence of non-human elements in the emergence of teachers' reflections. However, the application of such concepts in my own research has revealed the need to further refine them, in order to capture and characterise in a more detailed way the relations that arise between an online design and its users.

Consider for instance the concept of instrumentation process. In this research I have identified instrumentation processes that favour the emergence of mathematical reflections (see case 3, section 7.4.1). Such processes promote the development of teachers' mathematical knowledge. However, I also have found instrumentation processes that favour the emergence of didactical reflections (see case 4, section 7.4.2). That kind of processes facilitate the identification of values related to teachers' teaching practice. It is therefore necessary to refine the characterisation of the instrumentation processes according to the aspects of teachers' professional knowledge that they help to develop.

With regard to the concept of documentational orchestration, an idea that so far has only been implicitly considered is the cyclical or iterative nature of a documentational orchestration. Here I want to claim that, just like the documentational genesis, the documentational orchestration can be viewed as a process. A process in which an orchestration is applied and its application produces (or it does not produce) certain instrumentalization and instrumentation processes. Then taking into account these processes, the orchestration may be redesigned or transformed into a new orchestration.

The cyclical and evolutionary nature of a documentational orchestration allows to suggest that the long-term study of the documentational orchestrations used in teacher education institutions, could provide us with information regarding the development of teacher educators and the institutions themselves. For example, if we focus on the kind of orchestrations that a teacher educator uses for a given class of situations and observe them over a period of time, very likely we will detect changes in such orchestrations. The changes in the orchestrations may be linked to the development and changes that the teacher educator is experiencing through her practice. This is another area of research that could be explored.

8. Discussion of the research results

This last chapter of the dissertation presents the research results as a distillate from the analyses presented in the previous chapters. The chapter is divided into four sections:

- What are the research results?
- Are the research results reliable?
- Scope of the research results
- Implications of the research results

8.1 What are the research results?

This section presents a recapitulation of the research questions originally posed. It contains two sections in which the answers to the research questions are presented.

8.1.1 Answer to the first research question

The first research question posed was:

> **(1) What are the characteristics of the online interactions that promote the emergence of mathematics teachers' reflections?**

During the implementation of the first online course two cases of interactions that promoted the emergence of reflections were identified (see chapter 5). The interactions were characterised through the identification of the communicative acts that were present in them. Such interactions have several common communicative acts, namely: thinking aloud acts, getting in contact acts, locating acts, and evaluating acts. However, there are two characteristics that seem to be crucial to the emergence of reflections:

1. The (good) quality of the contact
2. The presence of evaluative and challenging acts

(1) The (good) quality of the contact. In order for a reflection to arise within an online interaction it is necessary that the participants of the interaction get into contact with each other. In other words, the ideas and opinions of all the participants in the interaction must be taken into account. This means that the participants of an online interaction must show a real interest in reading, analysing and trying to understand the ideas of their interlocutors.

When the contact between the participants of an online interaction is not mutual, it can foster a superficial interaction where not all the ideas that are present in the interaction are equally considered. In such situations valuable ideas may be disregarded. Neglected ideas typically offer different perspectives, new ways of interpreting a situation, which in turn can serve as a basis for the emergence of reflections.

(2) The presence of evaluative and challenging acts. The presence of evaluative and challenging acts within the online interactions is an attribute that is very important for the emergence of reflections. For example, the case 1 presented in chapter 8 illustrates how the evaluative acts from his colleagues helped a teacher to experience a mathematical reflection. Such reflection helped him to reconsider the way in which he was (mis)interpreting a graph.

The challenging and the evaluative acts force us to rethink the ideas we take for granted. Through those communicative acts we are able to see that there are alternative ways to interpret a situation. When our ideas, actions or values are assessed or challenged, one of the most basic mechanisms of reflection may be triggered; namely, the explicit consideration of our ideas and actions.

8.1.2 Answer to the second research question

The second research question posed was:

> **(2) Which non-human elements of an online course promote the emergence of mathematics teachers' reflections?**

During the implementation of the second online course two instances of reflections were detected (see chapter 7). However, only one of those instances was clearly triggered by a non-human resource. Here I refer to theoretical concepts from mathematics education research.

Here is an important issue that should be pointed out: When I use the expression "clearly triggered" I refer to resources whose influence on the emergence of reflections was identified through the theoretical lenses provided by the concepts used in the research. They are theory-based results. However, there are also resources that seem to influence the emergence of reflections, but that were not identified through the use of theoretical constructs. These resources were identified by observing the operation of the courses that are part of the research design of the investigation. An example of such resources is the time provided by the asynchronous discussion forums. This example is discussed in section 8.2.1. Thus, although this second type of results are not theory-based results but just plain observations on the functioning of the research design, they however have implications for the research on reflective thinking. This point will be further discussed at the end of the section 8.4.3.

Theoretical concepts as a trigger for mathematics teachers' reflections. Chapter 7 shows the case of a teacher who experienced a didactical reflection on how he used technology in his mathematics teaching (see section 7.4.2). The didactical reflection was clearly triggered by the concepts of pragmatic value and epistemic value presented in Lagrange (2005). Through this reflection the teacher discovered that he had a pragmatic perspective on the use of technology. That is, the teacher perceived the use of technology as a means for simplifying mathematical calculations and procedures, but not as a means for obtaining mathematical knowledge.

In chapter 7 it is argued that some theoretical concepts from mathematics education research may enable mathematics teachers to distance themselves from their own teaching practice and observe it from a different perspective. When mathematics teachers have the opportunity to observe their own practice from the perspective offered by mathematics education research (which is a sort of "outsider perspective"), the identification of values associated with their teaching practice is favoured. It would be more difficult to make visible such values without the distance provided by the concepts and ideas from mathematics education research.

8.2 Are the research results reliable?

In order to answer the question "are the research results reliable?" it is necessary to discuss the structure that allowed me to obtain the research results. It is necessary to explicitly discuss the effectiveness of such structure. In particular I will discuss the structure of the online courses that were designed; the nature of the empirical evidence presented in the dissertation; and the usefulness of the theoretical concepts that were used during the research process.

8.2.1 Discussion of the structure of the online courses

The discussion of the structure of the online courses will be centred in the effectiveness that they had to comply with their scientific aim.

In general, the scientific aim of the online courses was to promote the emergence of mathematics teachers' reflections, and to serve as a space where those reflections could be registered for subsequent analyses directed by the research questions. In the case of the first online course (the course on mathematical modelling), its role was also to promote interaction among teachers. Thus, the discussion on the effectiveness of the courses will be divided into three sections: (1) effectiveness in promoting interactions, (2) effectiveness in promoting reflections, and (3) effectiveness in registering instances of reflections.

(1) Effectiveness in promoting interactions. This part of the discussion applies only to the course on mathematical modelling. One of the main measures taken in order to promote interactions was to set up heterogeneous working groups. That is, groups in which their members had different opinions or views on the topic addressed.

Interactions that were very valuable for the research appeared during the application of the first online course. But the most important point here is that, it can be argued that such interactions were caused by the heterogeneity of the working groups, and that such heterogeneity was in turn provoked by the structure of the course.

A first example is the case 1 presented in the fifth chapter (see section 5.3.1). In this case the interaction within the working group was fed by the different interpretations of the graph 5, which was included in the first activity of the course (see figure 9, section 5.3.1). While Alberto believed that the graph 5 could actually represent a physical movement, Mariana and Susana did not share such idea. The constitution of this working group with members having different interpretations regarding the graph 5 was made possible by the design of the first activity. In particular, I refer to the decision to ask the teachers to solve individually the first activity and send me their answers by email before forming the working groups. This measure allowed me to identify the people who had different interpretation of the graphs, and then gathered them together in the same group.

Another example is the case 3 also introduced in chapter 5 (see section 5.3.3). In this case the interaction was fed by the different opinions about how to find the best airplane of the competition. In this case the variety of opinions about how to solve the problem was favoured by the change made to the activity "the paper airplane problem" taken from Lesh & Caylor (2007). I refer to the decision to remove the original request about making judgements about the accuracy of the paper airplanes and replace it with the more general question "Which one is the best airplane?". The inclusion of this question caused that the problem became more open favouring thus the emergence of different views on how the best aircraft should be selected.

The activity called "the marginalization index" was not effective for promoting interactions, though. Despite the fact that, the same strategy to form heterogeneous groups was applied (asking the teachers to solve individually the tasks before forming the groups), it was difficult to establish heterogeneous working groups. This was due to the fact that teachers' responses were very general, making difficult to locate specific opinions that could be confronted. For example, when teachers were asked "what is your opinion about the analysis of the ninth socio-economic indicator made by Emma, Carlos and Sandra", the answers typically obtained were like the following:

> "I find interesting the way they worked and the way in which they analysed the formula that assigned to them"
>
> "I think that the analysis done by Carlos and Emma is good"

In addition, several teachers expressed difficulties to address the second question of the activity in which they were asked to analyse by themselves one of the socio-economic indicators. For example one teacher expressed:

> "Dear teacher: I do not understand how to carry out the request included in paragraph 2. I do not have any idea about how to cope with the data. The tables are huge and they are already processed, I do not understand what you are asking us to do. I am sorry but I can not solve this point ..."

I think that the activity "the marginalization index" is a good example of the tension that existed between the scientific aim and the didactical aim of the activities included in the online courses. Probably this activity was not adequate enough to meet its scientific aim (to promote online interactions). However I think it was adequate to meet its didactical aim (to illustrate the application of mathematics in solving socially relevant problems).

I think however that the activity could be redesigned to better fulfil its scientific aim. For instance, before introducing the activity, we could ask teachers to express their views on the application of mathematics in society. That is, how they think mathematics is applied in the solution of social problems and what are the consequences of such application. Some teachers will probably have a perception of the application of mathematics as a "gentle and clean"[46] process. This is, its application is only related to

[46] I borrowed this term from Skovsmose (2005).

progress and human welfare. There are probably teachers with different perspectives and experiences regarding the application of mathematics, and then it would be possible to create heterogeneous groups to discuss the activity. If only teachers with a clean and gentle perception of the application of mathematics are identified, then I think that the activity itself could help to challenge such perception and to promote discussion and interaction.

(2) Effectiveness in promoting reflections. Several measures were implemented in the design of the online courses to try to promote teachers' reflections. Some of the measures were suggestions obtained from the literature, but there were also measures that were simply based on my previous experience as a teacher educator.

One example is the following. I assumed that gathering together teachers having different views and perspectives on a given problem or situation would favour the emergence of reflections. The case 1 included in the fifth chapter (and discussed in the previous section "(1) effectiveness in promoting interactions") is an example of how the variety of opinions and interpretations can serve as a basis for the emergence of reflections. When a teacher finds interpretations that are different from her own, this situation can contribute (although it does not guarantee) to making the teacher reconsider her own interpretations and in turn detonate the emergence of a reflection. Another example is the case 3 presented in chapter 7 (see section 7.4.1). It shows the case of a teacher who worked on the factorisation of the expression $x^n - 1$ in an algebraic context. When the teacher found the graphical perspective that one of his colleagues used to address the same problem, this detonated a mathematical reflection in which the teacher gave a new meaning to his own algebraic explorations.

So, I think that the diversity of opinions and perspectives tends to favour the emergence of reflections.

It is worth noticing that the emergence of different perspectives on a given problem was connected with the course design. For example, in the case 3 referred to in the paragraph above, the distribution of teachers into two sub-teams was crucial for the emergence of alternative perspectives on the general factorisation of $x^n - 1$. The teachers who worked with instrumented techniques obtained different factorisations to the ones obtained by the teachers that worked with paper-and-pencil techniques. In addition, the former teachers were in position to explore other aspects of the factorisation, such as the graphic perspective introduced by Rosa through the figures 21 and 22.

The two suggestions to promote the emergence of reflections that were obtained from the literature are: (1) to provide teachers with time to reflect; and (2) to promote the communication of ideas in a written form. Both suggestions were included in the design of the courses by designating the asynchronous forums as the primary means of communication and interaction in the courses.

There is evidence suggesting that the time provided by the discussion forums is a factor that favours the emergence of reflections. A good example is the case 3 presented in the fifth chapter. In that case, a teacher named Nadia tries to carry out the mathematization of the "paper airplane problem" before passing through the systematization stage. This means, to start performing mathematical calculations to find the best airplane, without having defined what characteristics should have the best airplane. This view is expressed for example in her utterance number [16]. However, three days after the teacher Nadia expresses a different perspective on her utterance number [18]. This utterance was interpreted as the outcome of a mathematical reflection. My point here is that the mathematical reflection emerged precisely during those three days. Examples like this one make me conclude that the discussion forums provide teachers with enough time to review the comments of their colleagues and their own, which in turn contributes to the emergence of reflections.

On the other hand, there are no indications that the written communication promotes the emergence of reflections. Of course some of the reflections that the teachers experienced were influenced by the messages of their colleagues, which were expressed in a written form. Nevertheless, in this research no evidence exists that the act of writing *by itself* encourages teachers to reflect on their ideas or actions. However, the absence of evidence does not mean that I am denying the existence of a possible relationship between the act of writing and the promotion of reflections.

(3) Effectiveness in registering instances of reflections. The online courses were a suitable space to register instances of reflections. The written communication in the courses was a key element to identify outcomes of reflections for subsequent analyses and for documenting the empirical basis for the conclusions drawn.

For instance, in the case 1 presented in chapter 5 (see section 5.3.1), a teacher called Alberto was discussing with his colleagues in an asynchronous forum the solution to the first activity of the modelling course. Before starting the discussion in the forum, teachers individually solved the activity and sent their written reports by email to me. The written reports allowed me to identify the initial interpretation that Alberto had of the graph 5 included in the first activity. Afterwards, in his utterance number [4] Alberto expressed a change of opinion about the graph 5. His initial interpretation had changed. My point here is that the written communication within the courses (the written reports, the discussion forums) allowed me to trace and identify outcomes of reflections. In the case of Alberto I was able to identify a positive change in his interpretation of the graph 5. I could also determine that the change in his interpretation was heavily influenced by the evaluative acts of his colleagues Mariana and Susana.

Another example that illustrates how written communication allowed me to identify and trace outcomes of reflections is the case 4 included in the seventh chapter (see section 7.4.2). In this case the discussion forum was a place where the pragmatic perspective that the teacher Francisco had on the use of technology in mathematics teaching was registered (see utterances [35] and [36]). However, the discussion forum also registered the outcome of the didactical reflection that Francisco experienced several days later. Here I refer to the utterances [38] and [39] where Francisco expressed that the concepts of pragmatic value and epistemic value helped him to identify his pragmatic stance on the use of technology in the teaching of mathematics.

Hence, I think that having privileged written communication and written interaction in the design of the courses was a very convenient and useful measure to register outcomes of reflection and trace their constitution.

8.2.2 On the empirical evidence presented in the dissertation

In the early stages of my research, when I began to plan the type of empirical data that I would include in my dissertation, I thought: "I will not include all the instances of reflections. I will only include the representative examples. The key examples". I soon discovered that this was an overly optimistic perspective. I learned that reflection is an elusive process. In this research only five cases where some kind of reflection is expressed were detected. Two cases were presented in the fifth chapter of the dissertation, and two cases were included in chapter 7. The remaining case concerns the following comment that a teacher expressed in an asynchronous forum, where the activity called "the marginalization index" was being discussed:

> [37]
> *Theme:* Re: Mathematical ambiguity?
> *From:* Alfredo
> *Date:* Friday, 18th of April 2008, 01:22
>
> […] Usually, social changes are not –I think– directly associated with mathematics. Although we know that they are involved for example, when we are talking about the economy of an entity, or the birth rate, and even when it comes to diseases like diabetes, cancer and AIDS.
>
> Such social issues made me remind the current problem regarding Mexican Petroleum Company (Pemex), which is on the national agenda and producing a social division [...] I confess that the topic appeals to me, even to work it as an activity [...] it is possible to get many statistical data on Pemex's position in world ranking [...] and about the importance that Pemex has on the Mexican economy, and thus make a mathematical model to analyse the risks that privatisation would lead […]

I interpret the utterance [37] as the outcome of an extra-mathematical reflection. I think that Alfredo has discovered that mathematics can also be utilised to shape social reality. In fact I think that the second paragraph of the utterance [37] suggests that the teacher was inspired by the marginalization index activity to design new activities in which socially relevant issues[47] could be discussed through mathematics. However, this case was not included in chapter 5 where the outcomes of the modelling course were analysed using the IC-Model of Alrø & Skovmose (2002). The reason why this case was excluded from the analysis is that the utterance [37] appeared in the discussion in an isolated way. That is, there is no evidence that the instance of reflection manifested in [37] was promoted by the interaction with other people who participated in the forum. It appeared spontaneously and out of nowhere. Therefore it was not possible to apply the IC-Model for analysing the interactional conditions that favoured its emergence. In fact, the utterance [37] was one of the signs that indicated to me that teachers' reflections could also be influenced by non-human elements included in the online courses.

Thus, with the exception of the utterance number [37], chapter 5 presents all the cases in which instances of reflection were identified. The chapter 5 also included the case 2 in which no reflection occurred. This was done in order to compare the communicative differences that existed between the interactions in which reflections appeared and the interactions in which no reflections appeared. The case 2 is representative of the kind of interactions in which no reflections appear. It is representative in the sense that it is a case in which teachers interact, they solve the task assigned to them, but no reflection occurs.

In chapter 7, where the outcomes of the course on the use of technology were analysed, all the cases in which an instance of reflection was detected were presented (the instrumentation processes illustrated through the

[47] In the utterance [37] Alfredo refers to the case of Pemex which is a state-owned oil company. During the period when the modelling course was applied, there was an intense national debate in Mexico, regarding the possible privatisation of the oil company.

cases 3 and 4). No case was excluded. The seventh chapter also included two examples of instrumentalization processes (cases 1 and 2). These cases illustrate how some of the components of the online course were appropriated by the mathematics teachers.

So, what I want to communicate in this section is that the instances of reflections presented in the dissertation are not a selection of the “best examples”. I presented the cases that I was able to detect by using the theoretical and methodological tools that I selected. However, this does not mean that the cases presented contain all the reflections that arose during the implementation of the two online courses. It is likely that some teachers had experienced a reflection without expressing it in discussion forums. Probably if I had used a different research method I could identified some of the reflections not expressed in the forums (perhaps applying questionnaires or carrying out interviews). However, as discussed in the section 3.4.3 of the third chapter, I intentionally chose to study only the interactions that occur spontaneously. I mean, unlike some other researchers, I did not explicitly ask teachers to reflect. Thus, in the dissertation I have provided the reader only with a few instances of teachers’ reflections. However, the reader can be sure that such instances of reflections are authentic and were manifested in a spontaneous way.

8.2.3 Usefulness of the theoretical concepts applied in the research

There are four main theoretical concepts used in this research:

1. The concept of reflection
2. The IC-Model
3. The concept of documentational orchestration
4. The instrumentation and instrumentalization processes

In this section the usefulness of each of these concepts in the development of the research will be discussed.

(1) The concept of reflection. Reflection is a theoretical concept which is central to this research because its role was to help to identify instances of reflections within the discussion forums. I think that its role was satisfactorily fulfilled. The characterisation of the concept of reflection allowed me to transform such a cognitive process into a researchable and identifiable entity within an online setting. Two elements of the definition of the concept were particularly important to make it operational: The inclusion of the "Aha! moment" in which something is discovered or revealed; and the characterisation of the types of reflections (didactical, mathematical, extra-mathematical).

The "Aha! moment" allowed me to point out the existence of a reflection within the empirical data. A reflection by itself can not be directly grasped, but it is possible to locate evidence of the existence of the "Aha! moments" which are the outcomes of a reflection. Such moments were associated with instances where a teacher changed her view on a particular situation, or expressed any surprise or discovery. This is the indirect way in which the reflections were identified.

The characterisation of the types of reflections allowed me to produce a fine-grained classification of the type of (relevant) reflections that a mathematics teacher can experience. However, I think that the concept of didactical reflection could be further refined. I think that it would be necessary to distinguish between didactical reflections and pedagogical reflections. A pedagogical reflection could be defined as the one in which the mathematics teacher consciously considers the teaching practice and the role of mathematical education at a general level. Involves a conscious consideration of the role and function of mathematics in students' education and of the general constraints for the teaching and learning of mathematics. A didactical reflection could be considered as a more particular and contextualised type of reflection. In this type of reflection the mathematics teacher considers the suitability and consequences of a particular kind of instruction aimed at addressing a particular mathematical topic, applied within a certain context or for a certain learner. Because in this study only one case of a didactical reflection (as originally defined in chapter 3) was identified, the characterisation of such concept was not further refined.

(2) The IC-Model. The IC-Model was used to characterise the interactions than contain a reflection. Although this model was originally developed based on observations of face-to-face interactions between students and mathematics teachers, it was possible to apply it in an online setting. As discussed in chapter 5, there are two characteristics that facilitated the application of the IC-Model in an online setting: (1) The communicative characteristics that define the IC-Model can be expressed and identified in verbal and written communication; and (2) the communicative characteristics of the IC-Model can be used to characterise human interactions regardless of the type of "students" and "teachers" who are involved in the interaction. In this research the "students" were in-service mathematics teachers.

However, there were some difficulties in implementing the IC-Model in the analysis of the online interactions. In particular I refer to the difficulty I had to distinguish between locating acts and identifying acts. There were utterances in the online interactions that could be classified as locating acts, but also as identifying acts. The way I addressed this situation was to make explicit my own interpretation of both communicative acts, and to use this interpretation during the data analysis. The details of this interpretation have been presented in chapter 5.

After analysing the empirical data, it appears to me that the getting in contact act included in the IC-Model needs further characterisation. That is, I think it is not enough to identify the getting in contact acts that are present within an interaction. It is also necessary to discuss the quality of the contact. I mean, one must investigate if the contact is mutual and continuous. When the contact is not mutual and not continuous, interactions that do not favour the emergence of reflections can be created.

This research has contributed to the development of the IC-Model in two ways. Firstly, it has been documented empirically that the range of applicability of the IC-Model is broad. I mean, it is a theoretical tool which allows us to characterise face-to-face interactions, but also online interactions. As I already have mentioned, the application of the tool does not depends on the type of students and teachers who are involved in the interaction. Secondly, it has been pointed out the specific aspects of the model that should be refined in order to improve its potential as an analytical for online settings. I refer to the need for further characterising some of the communicative acts of the model; and the necessity for continuing investigating how the technological tools may modify the nature of such acts.

(3) The concept of documentational orchestration. This concept was created to conceptualise the arrangement of resources with which mathematics teachers interact during an in-service course. The usefulness of this concept in the research lies in two aspects.

Firstly, the concept demanded me to make an a priori analysis of the role of each of the resources within the arrangement. In other words, it required me to explicitly state what kind of "effect" I was expecting to produce on the teachers through each of the resources. It also required me to order the set of resources, i.e., to explicitly locate their position within the arrangement. Such order facilitated the establishment of a connection between the set of resources and the set of "effects" produced on teachers by the resources.

Secondly, the documentational orchestration (DO) is a structure that allowed me to observe the process of creation of a document by a mathematics teacher from a micro level perspective. That is, instead of observing the dialectical relationship resource/document and its evolution over time, as suggested by Gueudet & Trouche (2009), the DO makes the researcher to focus on the micro-dynamics of the process of creation of a document. The DO produces a zoom-in on the documentational genesis to observe how the arrangement of a set of resources shapes teachers' ideas, but also how the teachers use and modify the resources during the establishment of a document. Such observation process is strongly linked to the instrumentation and instrumentalization processes discussed in the next section.

(4) The instrumentation and instrumentalization processes. These two processes helped me to establish relationships between the set of resources provided by the teacher educator (the DO) and the set of teachers' reflections.

The concept of instrumentation process was particularly important for addressing the second research question. This concept made me to reconstruct the process through which a reflection is produced. However, unlike the IC-Model, the reconstruction of an instrumentation process requires to ignore the communicative characteristics of the process. The reconstruction of the instrumentation processes required me to focus on locating the resources that were involved in the constitution of a reflection and to try to understand how such resources influenced the emergence of a particular reflection.

In principle the concept of instrumentalization process did not seem to contribute to the explanation of how teachers' reflections are shaped in an online setting. This because the application of the concept requires us to focus only on how the resources provided by the teacher educator are appropriated and/or modified by the teachers. However, the study of this kind of processes revealed that teachers' reflections can also be triggered by instrumentalization processes; i.e. the reflections can be triggered by elements created by the teachers themselves and that the teacher educator can not control. This information does not directly answer the second research question, but it allows us to get a glimpse of the complexity of the process of development of reflections in an online setting.

I think that the theoretical concepts applied in this research allowed me to get a general overview of teachers' reflections in an online setting. The concepts have allowed me to empirically demonstrate the existence of teachers' reflections in online settings. The application of the concepts have also contributed to the understanding of how such reflections can be promoted. We now have enhanced our knowledge of the interactional conditions that favour their appearance, and of the controllable elements in the design of an online course that promote their emergence.

8.3 Scope of the research results

In this section two points will be discussed. Firstly, I want to discuss if the two research questions have been fully answered by the results obtained through this research. Secondly, I will discuss whether or not the results of my research could be applied in a different context and under different conditions.

8.3.1 Have the research questions been fully answered?

I think that the research question 1 has been satisfactorily answered. I claim this because it has been possible to identify communicative characteristics that are common to the interactions that promote the emergence of reflections. But additionally, it has been shown that their importance not only lies in the fact that such characteristics are common to such interactions. The identified characteristics are also important because they actually influence the emergence of teachers' reflections. Here I particularly refer to evaluative acts and challenging acts.

I think that the answer to the research question 2 is not comprehensive or conclusive. I established a theoretical and methodological structure to try to identify the non-human elements which favour the emergence of reflections, but it was only possible to identify one of such elements; namely, theoretical concepts from mathematics teacher education research. There is empirical evidence that suggests that there are non-human elements of a different nature that also influence the emergence of reflections. For instance, the time provided by the discussion forums (this point was discussed in Section 8.2.1) or the activity called "the marginalization index" that was part of the modelling course (this point was discussed in Section 8.2.2). However, the theoretical and methodological structure used to address the second research question was not able to provide solid evidence to confirm these observations. So I think that the answer to the second research question only identifies a small proportion of all the non-human elements in an online setting that have the potential to trigger mathematics teachers' reflections.

8.3.2 Generalisability of the results

This research was developed in an online setting. The empirical data from where the research results were drawn are mainly composed by excerpts from asynchronous discussions. However, the obtained results are not context-dependent. Let me elaborate on this point:

It does not seem surprising that the answer to the research question 1 is not context-dependent. This is because a theoretical tool that is not medium-dependent (the IC-Model) was applied. I mean, the communicative characteristics of the IC-Model can be identified in verbal communication, but also in written communication. Thus, the type of communicative acts that were identified as crucial to the emergence of reflections in an online setting could also be crucial for the emergence of reflections in a face-to-face setting. There are no indications that the role played by the evaluative acts and the challenging acts is determined by some particular feature of the online setting. Nor are there indications that the role played by such communicative acts had been influenced by the kind of people who participated in the study (in-service mathematics teachers). My point here is that one might expect that these communicative acts also promote the emergence of reflections in face-to-face settings where other kinds of subjects involved in the teaching and learning of mathematics interact.

However, I think there is a condition that should be preserved in order to try to get the same results in a different context. I refer to the need to create an environment where different views on the same mathematical topic may converge. Such condition can be achieved through the implementation of open-ended mathematical problems (like "the paper airplane problem"), where different valid solutions to the problem and several opinions on how to tackle the problem could emerge.

The answer to the research question 2 is in a similar situation. It is not a context-dependent result. To answer the second question, I tried to identify the non-human elements that favour the emergence of mathematics teachers' reflections, but only one of such elements was identified: theoretical concepts from mathematics education research. This element can also be used in face-to-face settings. I mean, face-to-face courses in which mathematics teachers are introduced to the concepts and results produced by the mathematics education research. Such introduction would have the purpose of promoting the emergence of teachers' reflections.

Nevertheless, to apply this result in a different context it is necessary to preserve the following condition: the subject you want to experience a reflection should possess some sort of teaching experience. To claim this I rely on the empirical data analysed in this research, but also in the research results reported in Tsamir (2008). The empirical data to which I refer are included in the case 4 presented in chapter 7 (section 7.4.2). That case illustrates a didactical reflection experienced by a teacher called Francisco. The didactical reflection was triggered by the concepts of epistemic value and pragmatic value. Here it is important to note that Francisco's reflection was anchored in his own teaching practice. That is, Francisco consciously considered the way he was using technology in his mathematics teaching and compared it with the uses discussed in Lagrange (2005). Francisco's teaching practice served as a reference point for the emergence of the reflection.

Similarly, Tsamir (2008) reports a study in which mathematics teachers are introduced to the study of mathematics education theories. The study reports the case of two mathematics teachers called Tim (an in-service teacher) and Betty (a pre-service teacher) who experienced reflections triggered by the theoretical concepts they were introduced to. The reflections experienced by Tim and Betty had as a benchmark a mathematics lesson class that they led and in which they observed how a group of students solved some mathematical tasks (see pages 223–226). So I think that the combination of practical experience and theoretical knowledge is what makes possible the emergence of mathematics teachers' reflections.

The generalisability of the results of this research suggests that the online settings should be seriously considered as an area of empirical research on reflective thinking in both, online settings and face-to-face settings as well. I think it is possible to use the online settings for theorising about the role of reflection in teacher development, and the conditions that favour the emergence of reflections in both settings. I am not claiming that all the research results obtained in an online setting will be automatically applicable to face-to-face settings. But I am assuring that the online settings can provide us with a unique experimental space where the entity reflection can be accessed in a more direct and spontaneous manner. An experimental space as this one will allow us to improve our theoretical understanding of such entity in the general context of mathematics teacher education.

8.4 Implications of the research results

Finally the implications of the research results will be discussed. The discussion will be divided into three sections: (1) the contributions to the mathematics teacher education research, (2) the practical applications of the research results, and (3) the new questions that arise from this research.

8.4.1 Contributions to mathematics teacher education research

The results of this research contribute to the development of a sub-area within the field of mathematics teacher education research. The sub-area is defined by the intersection of two research trends, namely, reflective practice and online mathematics teacher education.

There are three main research contributions. Firstly, this research provides empirical evidence on the existence of different types of reflections that mathematics teachers might experience. I refer to the mathematical reflections, extra-mathematical reflections, and didactical reflections. As discussed in the third chapter of the dissertation (see section 3.4.2), researchers in mathematics teacher education put particular attention to the kind of reflections that are related to teachers' actions in the classroom. The results of my research may help to broaden this discussion. The results show that there are other types of reflections that are relevant to the professional development of mathematics teachers; such as reflections on their mathematical knowledge, reflections on the role of mathematics in society, and reflections on the values that guide their teaching practice.

Secondly, through this research, elements of an online course that contribute to the emergence of teachers' reflections have been identified. For instance, theoretical concepts from mathematics education research, group discussions in which a multiplicity of opinions is present and the time provided by the asynchronous discussion forums.

Finally, the research method used to identify teacher's reflections in an online setting can be considered as a another contribution to the field. The method includes a definition of the concept of reflection aimed at locating instances of reflections without forcing their emergence. That is, the definition make the researcher to focus on the outcomes of a reflection, and use them as evidence of the existence of such reflection. Thus, the research as shown that it is possible to create conditions within an online setting so that teachers' reflections could be manifested in a spontaneous way. Furthermore, the research has illustrated how the processes through which the reflections are constituted can be traced and reconstructed.

8.4.2 Practical applications of the research results

As discussed in the introduction, this research was motivated by the problems I have experienced in my practice as an mathematics teacher educator in an online setting. Fortunately, I think that some practical recommendations that will help to improve my practice as a teacher educator (and the practice of other teacher educators) can be drawn from this research. Such practical recommendations are related to the way in which teachers' reflections can be encouraged.

The presence of challenging acts and evaluating acts within interpersonal interactions tend to promote the emergence of reflections. Particularly in interactions where a variety of opinions and perspectives is present. Even though I think that the presence of challenging acts and evaluating acts in an interaction should not be "decreed" by the teacher educator, I believe that they can be promoted. One way to promote them is to encourage the presence of a multiplicity of opinions. It has been already mentioned that such multiplicity can be encouraged through the use of open-ended tasks in group discussions. In turn, the challenging acts and the evaluating acts could be promoted through the interventions of a teacher educator in the discussion. I mean, in a discussion the teacher educator can pose direct questions to the teachers such as "what do you think about the idea of teacher so-and-so?" or "do you agree with the comment of somebody-or-other?"

Another practical recommendation is the use of research papers or other items containing theoretical concepts or results from mathematics education research, in the development of mathematics teachers. As I discussed in the seventh chapter of the dissertation, I think that mathematics education research can provide teachers with a pair of glasses to observe their practice from a new perspective. In turn, this new perspective can create a basis for the emergence of reflections, which could help teachers to critically analyse their practice.

The theoretical concepts and results from mathematics education research should not necessarily be communicated through written products (such as research papers). There are different and more dynamic ways through which they can be communicated. Here I am particularly thinking on the use of videos. Through the use of video recordings it is possible to discuss and communicate the contents of research papers, or even to illustrate the operation of educational designs produced in the field of mathematics education research. Just to try to inspire the reader, I suggest two specific examples:

The first example is a video that discusses the contents of the article Jacobs (2010) called "Feminist pedagogy and mathematics". The video is part of a personal project aimed at spreading among the Spanish speaking population, results and theoretical ideas from mathematics education research. The video can be accessed through the link: http://bit.ly/aitKrP

Another example is a video that illustrates the application of an instructional design included in the research paper Tsamir (2001). . The purpose of the instructional design is to create a cognitive conflict in an individual. This video was used some years ago in the educational program from where the empirical data used in this research were obtained. The purpose of the video was to introduce mathematics teachers, in a more dynamic way, to the concept of cognitive conflict. The video can be accessed through the link: http://www.twitvid.com/2ILIJ

8.4.3 New questions that arise from this research

There are two issues arising from this research that I would like to further investigate.

The first issue refers to the use of theoretical concepts from mathematics education research in the development of in-service mathematics teachers. As discussed in chapter 7, there are some studies (including this one) that suggest that the study of concepts and theories from mathematics education research promotes reflections on our own values and practices as mathematics educators. However, it is not entirely clear what kind of concepts and theories are adequate to achieve such purpose. In chapter 7 I put forward a hypothesis regarding this issue: those theories or concepts that mathematics teachers perceive as applicable or related to their teaching practice, are the most suitable to promote the emergence of teachers' reflections. I would like to develop new research projects aimed at testing this hypothesis.

The second issue is related to the study of the instrumentalization processes that arise within an online orchestration. I find interesting to investigate the role that such processes can play in the development of mathematics teachers. Consider for example the case 2 presented in chapter 7 (see section 7.3.2). The case shows how a teacher used a YouTube video to explain a mathematical technique to one of her colleagues. Here I have several unanswered questions: what effects (if any) did the video produce in the person who watched it? Did the video help the observer to learn anything new? Did the video foster some kind of reflection in the individual? Unfortunately I have no empirical evidence that could allow me to answer these questions. However, the case 3, which was also presented in chapter 7, shows that the instrumentalization processes can actually positively affect the mathematics teachers' knowledge. In that case, the instrumentalization process originated by Rosa provided teacher Federico with a more robust understanding of the general factorisation of $x^n - 1$. In addition, the example presented in the section 7.6.4 of the seventh chapter, suggest that the long-term study of the instrumentalization processes can provide us with information about the development of teachers' use of computational tools. These observations make appealing to me the study of the instrumentalization processes and their relationship with the development of mathematics teachers.

There is another aspect related to teachers' reflections that, although it does not emerge from the results obtained in this research, I think it would be interesting and relevant to study. I refer to the actual effects or changes that reflections can provoke in the mathematics teachers' practice. Do reflections provoke real changes in the practice of mathematics teachers? This question is beyond the scope of this research. However, I think it would be a very interesting methodological challenge to try to make connections between the emergence of mathematics teachers' reflections and actual changes in their teaching practice.

I would like to close this section on the implications of the results of this research with a final thought. Based on the literature reviews that I have carried out, I can say that reflection is considered as a crucial component in the development of mathematics teachers. However, this research has shown that reflection is an elusive process. It is difficult to empirically identify instances of reflections. In this research only a few instances of reflections were identified. Even more difficult is to establish connections between the emergence of reflections and the factors that trigger them.

Such complexity, which is inherent to the study of teachers' reflections, indicates that we must work harder in the development of methods and theoretical constructs that could allow us to improve our understanding of how reflection processes are developed and promoted. The "plain observations" that were produced within this research and that were briefly mentioned in section 8.1.2, are outcomes of this research but they are not theory-based results. However, such observations could serve as a basis to hypothesise and theorise about the factors that promote the emergence of reflections. That is an implication of such kind of results. For instance, it is necessary to developed research methods which could enable us to produce empirical evidence confirming that time plays a important role in the emergence of reflections.

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