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Linearizability of n-linear Sirups

Héctor J. Hernández*
Centro de Estudios Tecnológicos
Instituto de Ingeniería y Tecnología
Universidad Autónoma de Cd. Juárez
Cd. Juárez, Chih., México CP 32320
hector@uacj.mx

Dongxing Tang
Lab for Logic and Databases
Dept. of Computer Science
New Mexico State University
Las Cruces, NM, USA 88003
dtang@cs.nmsu.edu

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Abstract

A linear program is easier to evaluate than a nonlinear program. Hence, given a recursive program, it is desirable to find an equivalent linear program. However, not all nonlinear programs are linearizable. Theoretically, an m-linear program is easier to evaluate than an n-linear program when $m < n$, since the derivation tree of the former one is of smaller arity than the derivation tree of the latter. Thus, when an n-linear program is not linearizable, we would like to find another, equivalent m-linear program with $m < n$.

In this paper, we consider two possibilities of linearizing n-linear sirups. First, we consider the equivalence between an n-linear sirup and its derivative or its general ZYT-linearization, which are linear programs. We show that the problem of determining whether an n-linear sirup is equivalent to its derivative or to its general ZYT-linearization is NP*-hard. We then give a tighter condition which is necessary and sufficient for testing those equivalences. The other possibility is to consider the equivalence between an n-linear sirup and another m-linear program, $m < n$, called its k-ZYT-linearization, where $k = n - m$. We also prove that the problem of determining whether an n-linear sirup is equivalent to its k-ZYT-linearization is* NP*-hard. Then, we present a tighter, exact condition for testing whether an n-linear sirup is equivalent to its k-ZYT-linearization. We do not know whether testing any of the above equivalences is decidable.*

Keywords: deductive databases, optimization, datalog programs, linearization, sirups (single recursive programs)

*H. J. Hernández is also affiliated with: Laboratory for Logic, Databases, and Advanced Programming of the Department of Computer Science of New Mexico State University.

1 Introduction

A linear Datalog program is easier to evaluate than a nonlinear program [Ullman & Van Gelder, 1986] because there is one path in every proof tree. For example, in [Ullman, 1989] an efficient Kleene's Algorithm is proposed for computing the Generalized Transitive Closure program. Therefore, given any arbitrary nonlinear Datalog program, we would like to find another linear program which is equivalent to the original one. Unfortunately, this problem is known to be undecidable in general [Gaifman *et al.*, 1993] [Saraiya, 1990].

The linearization of some bilinear Datalog programs has been studied. In particular, given a bilinear sirup [Kanellakis, 1987] $\wp$ of the form:

$$\begin{array}{l} \wp: \\ p(X_1,\ldots,X_n) \text{ :- } q(X_1,\ldots,X_n). \\ p(X_1,\ldots,X_n) \text{ :- } p(Y_1,\ldots,Y_n), p(Z_1,\ldots,Z_n), G. \end{array}$$

where G is a conjunction of extensional database (EDB) predicates, we want to know whether $\wp$ is equivalent to the following program $\wp'$:

$$\begin{array}{l} \wp': \\ p(X_1,\ldots,X_n) \text{ :- } q(X_1,\ldots,X_n). \\ p(X_1,\ldots,X_n) \text{ :- } q(Y_1,\ldots,Y_n), p(Z_1,\ldots,Z_n), G. \end{array}$$

If $\wp$ is equivalent to $\wp'$, it is said that $\wp$ is *ZYT-linearizable* [Zhang **et al.**, 1990]. It is shown in [Saraiya, 1990] that the problem of testing whether a bilinear sirup is ZYT-linearizable is undecidable; if G does not have repeated predicate names and some other conditions are satisfied, that problem becomes decidable [Zhang **et al.**, 1990]. Ramakrishnan *et al.* [Ramakrishnan *et al.*, 1993] considered a case of ZYT-linearization in which they allow repeated predicates in G. They proved that it is *NP*-hard to determine whether a bilinear sirup such as $\wp$ is ZYT-linearizable.

Notice that a bilinear sirup has two ZYT-linearizations, which are obtained by expanding the two recursive predicates in the recursive rule respectively. In this paper, we consider the ***derivative***, which is a linear program that includes these two linearizations of the original program. The derivative of the above program $\wp$ is the following program.

$\wp''$:
$p(X_1,\ldots,X_n) :- q(X_1,\ldots,X_n).$
$p(X_1,\ldots,X_n) :- q(Y_1,\ldots,Y_n), p(Z_1,\ldots,Z_n), G.$
$p(X_1,\ldots,X_n) :- p(Y_1,\ldots,Y_n), q(Z_1,\ldots,Z_n), G.$

If $\wp$ is equivalent to $\wp''$, $\wp$ is called *differentiable.*

Clearly, the derivative of a program is more general than its ZYT-linearization, in the sense that the derivative of a bilinear program contains its ZYT-linearization: when a program is not ZYT-linearizable, the derivative of a bilinear program can compute more facts than its ZYT-linearization. The following example shows this.

Example 1.1: Given the following bilinear sirup
$p(X,Y) :- e(X,Y).$
$p(X,Y) :- p(X,Z), p(Z,Y), a(X,Z), b(Z,Y).$
its ZYT-linearization and derivative are as follows:
$p(X,Y) :- e(X,Y).$
$p(X,Y) :- e(X,Z), p(Z,Y), a(X,Z), b(Z,Y).$
$p(X,Y) :- e(X,Y).$
$p(X,Y) :- p(X,Z), e(Z,Y), a(X,Z), b(Z,Y).$
$p(X,Y) :- e(X,Z), p(Z,Y), a(X,Z), b(Z,Y).$

Let us now consider the output of the above two linear programs with the following input:

$$\{e(1,2), e(2,3), e(3,4), a(1,2), a(1,3), b(2,3), b(3,4)\}.$$

It is not difficult to check that the derivative computes the fact $p(1,4)$, which is not in the output of the ZYT-linearization. That is, the derivative can produce more facts than the ZYT-linearization when the original program is not ZYT-linearizable. Thus, the derivative of a given bilinear program approximates the original program better than its ZYT-linearization does when the given bilinear program is not ZYT-linearizable. Later, we show that this bilinear sirup is not differentiable. □

Similar to the derivative of bilinear sirups, the derivative of an n-linear sirup consists of n linear rules which are obtained by expanding all but one of the recursive atoms in the recursive rule by the basis rule. However, when n is large, the number of rules in the derivative is large. Thus, the derivate is still not efficient to evaluate. Hence we consider other ways to linearize n-linear sirups.

One way is to consider the equivalence of the original program and a program, called its *general* ZYT-linearization, which is composed of the basis rule and the recursive rule with all the recursive predicates except one (arbitrarily chosen) substituted by the EDB predicate in the basis rule. The following example illustrates this.

Example 1.2: Consider the following program that finds all paths of odd length in e:
$p(X,Y) :- e(X,Y).$
$p(X,Y) :- p(X,U), p(U,V), p(V,Y).$

In Section 5, we show that this program is general ZYT-linearizable. That is, it is equivalent to the following linear program:
$p(X,Y) :- e(X,Y).$
$p(X,Y) :- e(X,U), e(U,V), p(V,Y).$ □

However, not every n-linear sirup is general ZYT-linearizable. Let us show such a program.

Example 1.3: Let $\wp$ be the following program:
$e(S) :- e_0(S).$
$e(S) :- e(T), e(U), e(V), e_1(S,T,U,V).$

We want to test whether this program is general ZYT-linearizable, that is, we want to check whether $\wp$ is equivalent to the following program:

$\wp_1$:
$e(S) :- e_0(S).$
$e(S) :- e_0(T), e_0(U), e(V), e_1(S,T,U,V).$

Let us now consider the following facts as input to $\wp$ and $\wp_1$: $\{e_0(1), e_0(2), e_0(3), e_0(4), e_0(5), e_0(6), e_0(7), e_1(8,2,3,4), e_1(9,5,6,7), e_1(10,1,8,9)\}$.

It is not difficult to check that the fact $e(10)$ is produced from $\wp$ but not from $\wp_1$. Hence, the original program is not general ZYT-linearizable. Note that this program is shown not to be linearizable in [Afrati & Cosmadakis, 1989]. □

If an n-linear sirup is not general ZYT-linearizable, theoretically, it may be interesting to consider whether it might be equivalent to its k-*ZYT-linearization*, which consists of the basis rule and one recursive rule obtained by replacing the first k $(1 \leq k < n)$ recursive predicates by the EDB predicate in the basis rule. If they are equivalent, we say that the original program is k-*ZYT-linearizable.* Thus, we might check whether $\wp$ in the previous example is 1-ZYT-linearizable, that is to check whether it is equivalent to the following program.

$\wp_2$:
$e(S) :- e_0(S).$
$e(S) :- e_0(T), e(U), e(V), e_1(S,T,U,V).$

It is clear that given a bilinear sirup, its 1-ZYT-linearization is its ZYT-linearization [Zhang *et al.*, 1990]. Moreover, when $k = n - 1$, then the k-ZYT-linearization is the general ZYT-linearization. Hence, both derivative and general ZYT-linearization generalize ZYT-linearization, while the k-ZYT-linearization is the generalization of general ZYT-linearization.

The rest of the paper is organized as follows: Section 2 gives relevant definitions and some simple facts on Cartesian products of relations used in the paper. Section 3 considers the differentiability of a bilinear sirup. Section 4 generalizes the results in Section 3 and considers the derivative of an n-linear program with $n \geq 3$. Section 5 generalizes ZYT-linearization to the case of an n-linear sirup for arbitrary n. Section 6 considers k-ZYT-linearization. Our conclusion is given in the last section.

2 Definitions and Basic Facts

This paper considers only Datalog programs, which in the sequel shall be referred to as *programs*. We assume the reader is familiar with standard terminology (see Ul88,Ul89), and only define some of the relevant, nonstandard notation used in the paper.

A *sirup* is a program which consists of one recursive rule and one base, nonrecursive rule [Kanellakis, 1987], called *basis* (or *initialization*) rule, such that both rules define the same intensional database (IDB) predicate. A sirup is *simple* if its basis rule has only one atom in its body, i.e., its basis rule is of the form $p(X_1, X_2, \ldots, X_n)$:- $q(X_1, X_2, \ldots, X_n)$. In this paper, we restrict ourselves to such simple programs. A sirup is *n-linear* if the predicate in the head of the recursive rule appears n times in its body.

A *proof tree* is a tree description for the derivation of an intensional fact by the application of some rules to extensional facts and the set of intensional facts generated earlier. The leaves of a proof tree must be EDB literals. The root, as well as any nonleaf node, must be an IDB literal. Every nonleaf node represents an application of a rule. The children of such a node are the subgoals in the body of the rule, instantiated to suitable EDB and IDB facts, of which the node is the head.

Given a program $\mathcal{P}$ and a database I, $\mathcal{P}(I)$ denotes the *output of* $\mathcal{P}$ *when* I *is its input*; that is, $\mathcal{P}(I)$ is the least fixed point of $\mathcal{P}$ with respect to I [Ullman, 1988]. If r is a rule, $r(I)$ shall denote the output of program { r } when I is its input. Given two programs $\mathcal{P}$ and $\mathcal{Q}$, $\mathcal{P}$ is *contained* in $\mathcal{Q}$, denoted $\mathcal{P} \subseteq \mathcal{Q}$, if over any database I, defined only on EDB predicates, $\mathcal{P}(I) \subseteq \mathcal{Q}(I)$; $\mathcal{P}$ is *contained in* $\mathcal{Q}$ *wrt predicate* q, denoted $\mathcal{P} \subseteq_q \mathcal{Q}$, if over any database I, defined only on EDB predicates, every fact t defined over predicate q in $\mathcal{P}(I)$ is also in $\mathcal{Q}(I)$. $\mathcal{P}$ is equivalent to $\mathcal{Q}$, written $\mathcal{P} \equiv \mathcal{Q}$, if $\mathcal{P} \subseteq \mathcal{Q}$ and $\mathcal{Q} \subseteq \mathcal{P}$. $\mathcal{P}$ is *equivalent to* $\mathcal{Q}$ *wrt predicate* q, written $\mathcal{P} \equiv_q \mathcal{Q}$, if $\mathcal{P} \subseteq_q \mathcal{Q}$ and $\mathcal{Q} \subseteq_q \mathcal{P}$. Note that the above definition of equivalence of two programs is over extensional databases. If we allow both extensional and intensional databases as input, the equivalence is called *uniform equivalence*; $\mathcal{P} \equiv^u \mathcal{Q}$ denotes that $\mathcal{P}$ is uniformly equivalent to $\mathcal{Q}$. The problem of testing equivalence of two programs is undecidable [Shmueli, 1987], while the problem of testing uniform equivalence of two programs is decidable [Sagiv, 1987].

2.1 Graph G and rule r_G

In order to prove NP-hardness of the linearization problems studied here, we first show an important result from [Kanellakis, 1987]. Let G be a graph. Kanellakis has constructed the following rule from the graph G.

$$r_G : \; q(X,Y,Z,V) \text{ :- } q(X,Y,Z,W), \varphi_G(W,V)$$

r_G contains a number of literals for representing the graph; in particular, φ_G, which is a conjunction of EDB literals with occurrences of V and W, is used for that purpose; in this paper, without loss of generality, we assume that φ_G is an EDB atom.

We say that r_G is *1-bounded* if for any I, where I is a set of atoms defined on q and φ_G, $r_G(I)$ = $r_G^1(I)$, where the equality is restricted on q-facts, and $r_G^1(I)$, the nonrecursive application of r_G to I, is $r_G^1(I) = \{q(a_1,a_2,a_3,a_4) | \; q(a_1,a_2,a_3,a_4) \in I \vee \exists b_1(q(a_1,a_2,a_3,b_1) \in I \wedge \varphi_G(b_1,a_4) \in I)\}$.

Kanellakis proved the following result (Theorem 2 in [Kanellakis, 1987]) that we shall use later in our proofs.

Theorem 2.1: G is three-colorable iff r_G is 1-bounded.

2.2 Some Results on Cartesian Products

We now state some simple results about Cartesian products of relations that are needed later on in the paper. Before doing that, we need to define the following. Let r be a relation. Then, we define

- $r^0 = \{\epsilon\}$,where ϵ is the empty tuple such that $r' \times \{\epsilon\} = \{\epsilon\} \times r' = r'$ for any relation r'; and
- $r^n = r \times r^{n-1}$, if $n \geq 1$.

Lemma 2.1: Let r be a relation, let r_1 be a subset of r, and assume that r and r_1 are not empty. Then for any n, $n \geq 2$, $(r \times r_1^{n-1}) \cup (r_1 \times r \times r_1^{n-2}) \cup \cdots \cup (r_1^{n-1} \times r) = r^n$ iff $r_1 = r$.

Proof: The *if*-part is obvious. For the other direction, let us suppose that $r \neq r_1$. Then let t be a tuple in $r - r_1$. It is not difficult to see that t', the tuple in $(\{t\})^n$, is not a member of

$$(r \times r_1^{n-1}) \cup (r_1 \times r \times r_1^{n-2}) \cup \cdots \cup (r_1^{n-1} \times r) \qquad (2)$$

because each of the terms in (2) has $n-1$ occurrences of r_1, and t does not belong in r_1. □

3 Differentiability of bilinear sirups

It is shown in [Ramakrishnan *et al.*, 1993] that the problem of deciding whether a bilinear sirup is ZYT-linearizable is NP-hard. In this section, we prove that the problem of deciding if a bilinear sirup is differentiable is also NP-hard; we do not know if it is decidable. In addition, we show a necessary and sufficient condition of differentiability of bilinear sirups.

$q(Z'_1,\ldots,Z'_n), G', s(Z_1,\ldots,Z_n), G.$
$r''_5 : p(X_1,\ldots,X_n) \text{ :- } s(Y_1,\ldots,Y_n),$
$q(Y''_1,\ldots,Y''_n), q(Z''_1,\ldots,Z''_n), G'', G.$

$\wp_3$:
$r'''_0 : p(X_1,\ldots,X_n) \text{ :- } q(X_1,\ldots,X_n).$
$r'''_1 : p(X_1,\ldots,X_n) \text{ :- } q(Y_1,\ldots,Y_n),$
$p(Z_1,\ldots,Z_n), G.$
$r'''_2 : p(X_1,\ldots,X_n) \text{ :- } p(Y_1,\ldots,Y_n),$
$q(Z_1,\ldots,Z_n), G.$
$r'''_3 : p(X_1,\ldots,X_n) \text{ :- } q(Y'_1,\ldots,Y'_n),$
$q(Z'_1,\ldots,Z'_n), G', p(Z_1,\ldots,Z_n), G.$
$r'''_4 : p(X_1,\ldots,X_n) \text{ :- } p(Y_1,\ldots,Y_n),$
$q(Y''_1,\ldots,Y''_n), q(Z''_1,\ldots,Z''_n), G'', G.$

$\wp_3$ is obtained from $\wp_2$ by substituting predicate p for predicate s; $\wp_2$ consists of $\wp_1$ with the recursive predicate p replaced by s submitted to p by rule r''_3 plus the rules r''_4 and r''_5; r''_4 is obtained by expanding the first recursive atom in the body of r_1 by r_1 itself and then substituting the first two recursive predicates p in the expanded rule with the exit predicate q and the last recursive predicate p with s; G' is G after the expansion of the first recursive atom of r_1; r''_5 is obtained similarly by expanding the second recursive atom in the body of rule r_1 in $\wp$ and then substituting the last two predicates p with the exit EDB predicate q and the first recursive predicate p with s; G'' denotes G after the expansion in which we substitute the second recursive atom of r_1.

In the following theorem, we show that the problem of differentiability of $\wp$ may be reduced to the problem of testing the equivalence of two linear programs.

Theorem 3.2: Let programs $\wp$, $\wp_1$, and $\wp_2$ be as defined above. Then, $\wp$ is differentiable if and only if $\wp_1 \equiv_p \wp_2$.

Proof: Clearly, $\wp_1 \subseteq_p \wp_2 \subseteq_p \wp$. Thus, if $\wp$ is differentiable, we have $\wp \subseteq \wp_1$. Hence, $\wp_1 \equiv_p \wp_2$.

Now we prove the other direction. Assume that $\wp_1 \equiv_p \wp_2$. We shall prove by induction on the height of a proof tree that every proof tree which derives a fact t from $\wp$ has a proof tree from $\wp_1$ which also derives that fact. For the proof tree with height 1, it is trivial (because proof trees for facts produced by r_0 are proof trees for facts produced by r'_0).

For the induction, we assume that a fact derived from a proof tree with height less than h, $h > 1$, from $\wp$ can also be derived by a proof tree from $\wp_1$.

We now consider a proof tree T from $\wp$, which derives some fact $p(x)$, and assume that its height is equal to h. Since $\wp$ is bilinear, for each interior node $p(v)$ of T that has two p subgoals as children at the second last level, as shown in Figure 1(a), we compact it into an EDB atom $q(v)$ as a leaf as shown in Figure 1(b); we assume that we mark this node somehow, since we need to uncompact it later; the above x and v are constant vectors. According to the modifications we made to T, we change the database from an original relation Q to some new relation Q' which contains these compacted tuples.

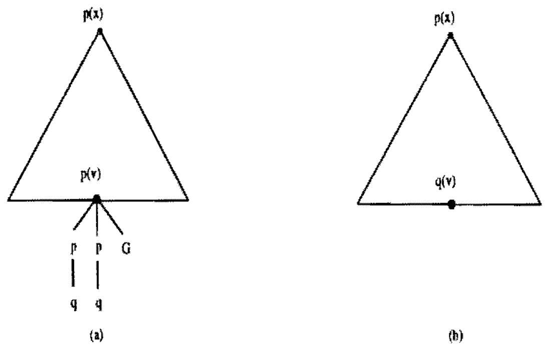

Figure 1: Reducing the height of a proof tree

By the inductive hypothesis, there exists a proof tree as shown in Figure 2(a), which derives the same tuple in $\wp_1$ from the new relation Q'; the proof tree of the derivative of a bilinear program looks like a zigzag (one in which either left-hand or right-hand recursive nodes are expanded by the basis rule). We next uncompact all the tuples $q(v)$, which we marked above to distinguish them, in the proof tree and return to the original database as shown in Figure 2(b); let us use T' to denote the proof tree obtained in this step. We now prove that for each sub-proof tree like the one rooted at p_i on Figure 2(b) there is a proof tree in $\wp_2$ that computes p_i, and hence a proof-tree in $\wp_1$ (since $\wp_2 \subseteq_p \wp_1$). This shall prove that we can transform a tree like T' into a proof tree from $\wp_1$.

We walk up T', beginning at the bottom until we find the first interior node p_i with either a left or right branch that came from a compact node. From the proof tree rooted at p_i, we can obtain a proof tree in $\wp_2$ as shown in Figure 2(c) that derives the same fact p_i: the leftmost subtree is the counterpart of the leftmost subtree in the proof tree shown in the box of Figure 2(b), where we replaced the predicate p by s; the q-facts and G and G' are the same as in the proof tree shown in the box; the proof tree rooted at p_i in Figure 2(c) represents an application of rule r''_5 at the top. Since $\wp_2 \subseteq_p \wp_1$, we have a proof tree from $\wp_1$ which produces the same fact p_i. The other kind of sub-proof tree transformation requires the application of rule r''_4 instead of r''_5 as above. This completes the proof of this part of the theorem and the proof of the theorem itself. □

Note that in fact when we test if a bilinear program is differentiable, we can directly test if its derivative is equivalent to the original one. However, Theorem 3.2 shows us a tighter condition since both $\wp_1$ and $\wp_2$ are linear. We next use $\wp_3$ to give a condition to test the differentiability of $\wp$.

Theorem 3.3: Let $\wp$, $\wp_1$, and $\wp_3$ be as defined previously. Then $\wp$ is differentiable if and only if $\wp_1$ is equivalent to $\wp_3$.

$$\begin{aligned}
&\qquad q(Z'_1,\ldots,Z'_n), G', s(Z_1,\ldots,Z_n), G.\\
&r''_5 : p(X_1,\ldots,X_n) \mathrel{:\!-} s(Y_1,\ldots,Y_n),\\
&\qquad q(Y''_1,\ldots,Y''_n), q(Z''_1,\ldots,Z''_n), G'', G.
\end{aligned}$$

$\wp_3$:

$$\begin{aligned}
&r'''_0 : p(X_1,\ldots,X_n) \mathrel{:\!-} q(X_1,\ldots,X_n).\\
&r'''_1 : p(X_1,\ldots,X_n) \mathrel{:\!-} q(Y_1,\ldots,Y_n),\\
&\qquad p(Z_1,\ldots,Z_n), G.\\
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&r'''_4 : p(X_1,\ldots,X_n) \mathrel{:\!-} p(Y_1,\ldots,Y_n),\\
&\qquad q(Y''_1,\ldots,Y''_n), q(Z''_1,\ldots,Z''_n), G'', G.
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$\wp_3$ is obtained from $\wp_2$ by substituting predicate p for predicate s; $\wp_2$ consists of $\wp_1$ with the recursive predicate p replaced by s submitted to p by rule r''_3 plus the rules r''_4 and r''_5; r''_4 is obtained by expanding the first recursive atom in the body of r_1 by r_1 itself and then substituting the first two recursive predicates p in the expanded rule with the exit predicate q and the last recursive predicate p with s; G' is G after the expansion of the first recursive atom of r_1; r''_5 is obtained similarly by expanding the second recursive atom in the body of rule r_1 in $\wp$ and then substituting the last two predicates p with the exit EDB predicate q and the first recursive predicate p with s; G'' denotes G after the expansion in which we substitute the second recursive atom of r_1.

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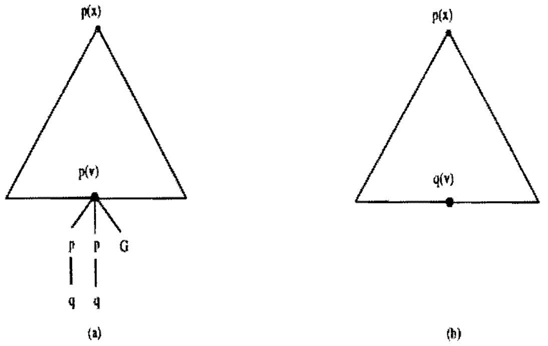

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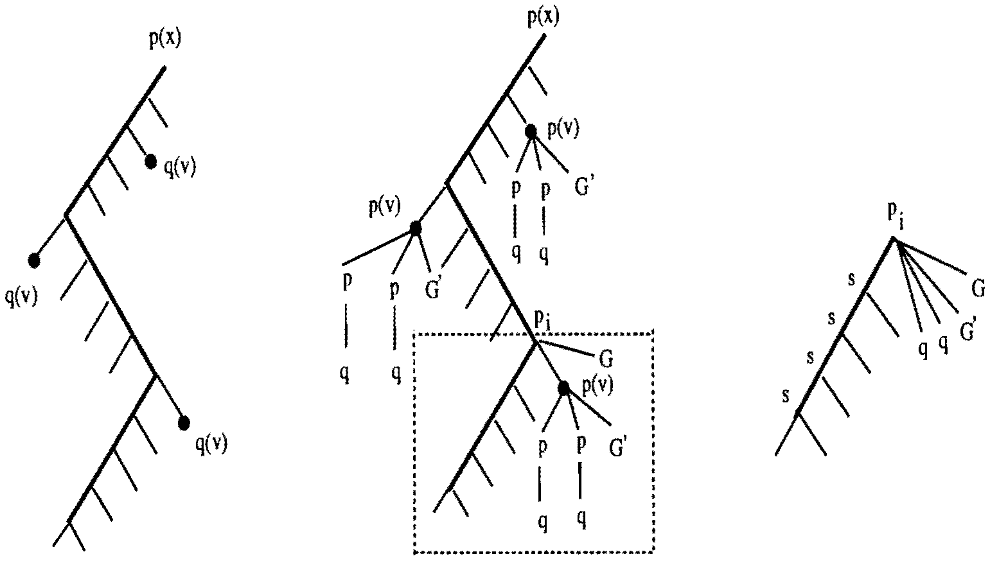
We walk up T', beginning at the bottom until we find the first interior node p_i with either a left or right branch that came from a compact node. From the proof tree rooted at p_i, we can obtain a proof tree in $\wp_2$ as shown in Figure 2(c) that derives the same fact p_i: the leftmost subtree is the counterpart of the leftmost subtree in the proof tree shown in the box of Figure 2(b), where we replaced the predicate p by s; the q-facts and G and G' are the same as in the proof tree shown in the box; the proof tree rooted at p_i in Figure 2(c) represents an application of rule r''_5 at the top. Since $\wp_2 \subseteq_p \wp_1$, we have a proof tree from $\wp_1$ which produces the same fact p_i. The other kind of sub-proof tree transformation requires the application of rule r''_4 instead of r''_5 as above. This completes the proof of this part of the theorem and the proof of the theorem itself. □

Note that in fact when we test if a bilinear program is differentiable, we can directly test if its derivative is equivalent to the original one. However, Theorem 3.2 shows us a tighter condition since both $\wp_1$ and $\wp_2$ are linear. We next use $\wp_3$ to give a condition to test the differentiability of $\wp$.

Theorem 3.3: Let $\wp$, $\wp_1$, and $\wp_3$ be as defined previously. Then $\wp$ is differentiable if and only if $\wp_1$ is equivalent to $\wp_3$.



(a) $\wp_1$'s proof tree (b) Uncompacted tree T' (c) A proof tree from $\wp_2$

Figure 2: Proof tree restoration and sub-proof tree transformation

Proof: It is easy to see that $\wp_1 \subseteq_p \wp_2 \subseteq_p \wp_3 \subseteq_p \wp$. Thus, the *only-if* part is trivial. The *if* part follows from Theorem 3.2. □

The problem of testing whether two Datalog programs are equivalent is undecidable [Shmueli, 1987]. Hence, the tests given by both Theorem 3.2 and Theorem 3.3 are noneffective. Fortunately, uniform equivalence is decidable [Sagiv, 1987] and uniform equivalence implies equivalence. Moreover, by Sagiv [Sagiv, 1987], to check if $\wp_3$ is uniformly contained in $\wp_1$ suffices to check if r_3''' and r_4''' are contained in $\wp_1$ individually. Since $\wp_1$ is a linear program, when r_3''' or r_4''' is considered as a conjunctive query and its body is viewed as a database input to $\wp_1$, we can search for the derivations of $\wp_1$ using a nondeterministic polynomial space algorithm [Chandra *et al.*, 1981]. Following the above discussion, we have the following theorem.

Theorem 3.4: Let $\wp$, $\wp_1$, and $\wp_3$ be as defined previously. Then, $\wp$ is differentiable if $\wp_1$ is uniformly equivalent to $\wp_3$. This condition can be tested in polynomial space.

Example 3.5: Consider the following program:

$\wp$:
$r_0 : p(X,Y) \text{ :- } e(X,Y).$
$r_1 : p((X,Y) \text{ :- } p(X,Z), p(Z,Y), a(X,Z), b(Z,Y).$

We want to test whether this program is differentiable. It is well-known that the bilinear version of the transitive closure problem (i.e., the recursive rule r_1 in $\wp$ without $a(X,Z)$ and $b(Z,Y)$) is equivalent to its ZYT-linearization, hence it is differentiable. However, we now show that the above program $\wp$ is not differentiable. Note that the derivative of $\wp$ is as follows:

$\wp_1$:
$r_0' : p(X,Y) \text{ :- } e(X,Y).$
$r_1' : p(X,Y) \text{ :- } p(X,Z), e(Z,Y), a(X,Z), b(Z,Y).$
$r_2' : p(X,Y) \text{ :- } e(X,Z), p(Z,Y), a(X,Z), b(Z,Y).$

To test whether $\wp$ is differentiable, by Theorem 3.3, we need to check whether $\wp_1$ is equivalent to $\wp_3$ which consists of $\wp_1$ and rules r_3' and r_4':

$\wp_3$:
$r_0' : p(X,Y) \text{ :- } e(X,Y).$
$r_1' : p(X,Y) \text{ :- } p(X,Z), e(Z,Y), a(X,Z), b(Z,Y).$
$r_2' : p(X,Y) \text{ :- } e(X,Z), p(Z,Y), a(X,Z), b(Z,Y).$
$r_3' : p(X,Y) \text{ :- } p(X,Z), e(Z,Z_1), e(Z_1,Y), a(Z,Z_1), b(Z_1,Y), a(X,Z), b(Z,Y).$
$r_4' : p(X,Y) \text{ :- } e(X,Z_1), e(Z_1,Z), a(X,Z_1), b(Z_1,Z), p(Z,Y), a(X,Z), b(Z,Y).$

Let us now consider the following set of facts as input:

$$\{e(1,2), e(2,3), e(3,4), e(4,5), a(1,2), a(3,4), a(1,3), b(2,3), b(4,5), b(3,5)\}$$

We can check that $\wp_1$ computes the following p-facts:

$$\{p(1,2), p(2,3), p(3,4), p(4,5), p(1,3), p(3,5)\}$$

while $\wp_3$ computes p-facts as follows:

$$\{p(1,2), p(2,3), p(3,4), p(4,5), p(1,3), p(3,5), p(1,5)\}$$

Note the difference $p(1,5)$ between the two outputs. Thus, $\wp_3$ is not contained in $\wp_1$. Therefore, by Theorem 3.3, we conclude that the original program $\wp$ is not differentiable. □

4 Differentiability of n-linear sirups

In the preceding section, we concentrated on bilinear programs. We have shown that it is *NP*-hard to determine whether a bilinear program is differentiable and we have also given a necessary and sufficient condition of its differentiability. In this section, we consider n-linear programs for arbitrary n. We prove that the problem of deciding whether an n-linear sirup is differentiable is also *NP*-hard. In addition, we show a necessary and sufficient condition of differentiability of an n-linear sirup.

Theorem 4.1: It is *NP*-hard to decide whether an n-linear sirup is differentiable.

Proof: The proof idea is identical to the one in Theorem 3.1, except that we use the following n-linear program $\wp$ in which the recursive predicate has $4n$ arguments; its derivative $\wp_1$ is also shown below.

$\wp$:

$r_0 :\ p(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$X_{n1}, X_{n2}, X_{n3}, X_{n4})$:-
$p_0(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$X_{n1}, X_{n2}, X_{n3}, X_{n4})$.

$r_1 :\ p(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$X_{n1}, X_{n2}, X_{n3}, X_{n4})$:-
$p(X_{11}, X_{12}, X_{13}, Y_{14}, \ldots, -, -, -, -)$,
$\varphi_G(Y_{14}, X_{14})$,
$p(X_{21}, X_{22}, X_{23}, Y_{24}, \ldots, -, -, -, -)$,
$\varphi_G(Y_{24}, X_{24})$,
$\ldots$,
$p(X_{n1}, X_{n2}, X_{n3}, Y_{n4}, \ldots, -, -, -, -)$,
$\varphi_G(Y_{n4}, X_{n4})$.

$\wp_1$:

$r'_0 :\ p(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$X_{n1}, X_{n2}, X_{n3}, X_{n4})$:-
$p_0(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$X_{n1}, X_{n2}, X_{n3}, X_{n4})$.

$r'_1 :\ p(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$X_{n1}, X_{n2}, X_{n3}, X_{n4})$:-
$p(X_{11}, X_{12}, X_{13}, Y_{14}, \ldots, -, -, -, -)$,
$\varphi_G(Y_{14}, X_{14})$,
$p_0(X_{21}, X_{22}, X_{23}, Y_{24}, \ldots, -, -, -, -)$,
$\varphi_G(Y_{24}, X_{24})$,
$\ldots$,
$p_0(X_{n1}, X_{n2}, X_{n3}, Y_{n4}, \ldots, -, -, -, -)$,
$\varphi_G(Y_{n4}, X_{n4})$.

$r'_2 :\ p(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$X_{n1}, X_{n2}, X_{n3}, X_{n4})$:-
$p_0(X_{11}, X_{12}, X_{13}, Y_{14}, \ldots, -, -, -, -)$,
$\varphi_G(Y_{14}, X_{14})$,
$p(X_{21}, X_{22}, X_{23}, Y_{24}, \ldots, -, -, -, -)$,
$\varphi_G(Y_{24}, X_{24})$,
$p_0(X_{31}, X_{32}, X_{33}, Y_{34}, \ldots, -, -, -, -)$,
$\varphi_G(Y_{34}, X_{34})$,
$\ldots$,
$p_0(X_{n1}, X_{n2}, X_{n3}, Y_{n4}, \ldots, -, -, -, -)$,
$\varphi_G(Y_{n4}, X_{n4})$.

$\vdots$

$r'_n :\ p(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$X_{n1}, X_{n2}, X_{n3}, X_{n4})$:-
$p_0(X_{11}, X_{12}, X_{13}, Y_{14}, \ldots, -, -, -, -)$,
$\varphi_G(Y_{14}, X_{14})$,
$p_0(X_{21}, X_{22}, X_{23}, Y_{24}, \ldots, -, -, -, -)$,
$\varphi_G(Y_{24}, X_{24})$,
$\ldots$,
$p_0(X_{(n-1)1}, X_{(n-1)2}, X_{(n-1)3}, Y_{(n-1)4},$
$\ldots, -, -, -, -), \varphi_G(Y_{(n-1)4}, X_{(n-1)4})$,
$p(X_{n1}, X_{n2}, X_{n3}, Y_{n4}, \ldots, -, -, -, -)$,
$\varphi_G(Y_{n4}, X_{n4})$.

The interested reader is referred to [Tang, 1996] for the details of the proof. □

4.1 Additional Conditions for Differentiability of trilinear sirups

Theorem 4.1 establishes a lower bound in the complexity of differentiability of an n-linear sirup. As in the case for bilinear sirups, given an n-linear program $\wp$, there are linear programs $\wp_2$ and $\wp_3$ such that $\wp$ is differentiable, i.e., equivalent to its derivative $\wp_1$, if and only if $\wp_1 \equiv_p \wp_2$ or $\wp_1 \equiv \wp_3$. Those programs are shown next; G is a conjunction of EDB literals. Theorems 3.2, 3.3, and 3.4 hold for these programs [Tang, 1996].

$\wp$:

$r_0 :\ p(\vec{X})$:- $q(\vec{X})$.
$r_1 :\ p(\vec{X})$:- $p(\vec{Y}), p(\vec{Z}), p(\vec{U}), G$.

$\wp_1$:

$r'_0 : p(\vec{X})$:- $q(\vec{X})$.
$r'_1 : p(\vec{X})$:- $p(\vec{Y}), q(\vec{Z}), q(\vec{U}), G$.
$r'_2 : p(\vec{X})$:- $q(\vec{Y}), p(\vec{Z}), q(\vec{U}), G$.
$r'_3 : p(\vec{X})$:- $q(\vec{Y}), q(\vec{Z}), p(\vec{U}), G$.

$\wp_2$:

$r''_0 : s(\vec{X})$:- $q(\vec{X})$.
$r''_1 : s(\vec{X})$:- $s(\vec{Y}), q(\vec{Z}), q(\vec{U}), G$.
$r''_2 : s(\vec{X})$:- $q(\vec{Y}), s(\vec{Z}), q(\vec{U}), G$.
$r''_3 : s(\vec{X})$:- $q(\vec{Y}), q(\vec{Z}), s(\vec{U}), G$.
$r''_4 : p(\vec{X})$:- $s(\vec{X})$.
$r''_{51} : p(\vec{X})$:- $s(\vec{Y}), q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G', q(\vec{U}), G$.
$r''_{52} : p(\vec{X})$:- $s(\vec{Y}), q(\vec{Z}), q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G', G$.
$r''_{53} : p(\vec{X})$:- $s(\vec{Y}), q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G',$
$q(\vec{Y''}), q(\vec{Z''}), q(\vec{U''}), G'', G$.
$r''_{61} : p(\vec{X})$:- $q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G', s(\vec{Z}), q(\vec{U}), G$.
$r''_{62} : p(\vec{X})$:- $q(\vec{Y}), s(\vec{Z}), q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G', G$.
$r''_{63} : p(\vec{X})$:- $q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G',$
$s(\vec{Z}), q(\vec{Y''}), q(\vec{Z''}), q(\vec{U''}), G'', G$.
$r''_{71} : p(\vec{X})$:- $q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G', q(\vec{Z}), s(\vec{U}), G$.
$r''_{72} : p(\vec{X})$:- $q(\vec{Y}), q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G', s(\vec{U}), G$.
$r''_{73} : p(\vec{X})$:- $q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G',$
$q(\vec{Y''}), q(\vec{Z''}), q(\vec{U''}), G'', s(\vec{U}), G$.

$\wp_3$:

$r'''_0 : p(\vec{X})$:- $q(\vec{X})$.
$r'''_1 : p(\vec{X})$:- $p(\vec{Y}), q(\vec{Z}), q(\vec{U}), G$.
$r'''_2 : p(\vec{X})$:- $q(\vec{Y}), p(\vec{Z}), q(\vec{U}), G$.
$r'''_3 : p(\vec{X})$:- $q(\vec{Y}), q(\vec{Z}), p(\vec{U}), G$.
$r'''_4 : p(\vec{X})$:- $p(\vec{Y}), q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G', q(\vec{U}), G$.
$r'''_5 : p(\vec{X})$:- $p(\vec{Y}), q(\vec{Z}), q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G', G$.
$r'''_6 : p(\vec{X})$:- $p(\vec{Y}), q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G',$
$q(\vec{Y''}), q(\vec{Z''}), q(\vec{U''}), G'', G$.
$r'''_7 : p(\vec{X})$:- $q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G', p(\vec{Z}), q(\vec{U}), G$.
$r'''_8 : p(\vec{X})$:- $q(\vec{Y}), p(\vec{Z}), q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G', G$.
$r'''_9 : p(\vec{X})$:- $q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G',$

$p(\vec{Z}), q(\vec{Y''}), q(\vec{Z''}), q(\vec{U''}), G'', G.$
$r'''_{10} : p(\vec{X}) \text{ :- } q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G', q(\vec{Z}), p(\vec{U}), G.$
$r'''_{11} : p(\vec{X}) \text{ :- } q(\vec{Y}), q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G', p(\vec{U}), G.$
$r'''_{12} : p(\vec{X}) \text{ :- } q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G',$
$q(\vec{Y''}), q(\vec{Z''}), q(\vec{U''}), G'', p(\vec{U}), G.$

Where: $\wp_3$ is a program obtained from $\wp_2$ by substituting predicate p for predicate s; r''_{51} is obtained by first expanding the second recursive atom p once in the body of the recursive rule in $\wp$ and then substituting the last four recursive predicates p in the expanded rule with the exit predicate q and the first p predicate with s; r''_{52} is obtained by expanding the last recursive predicate p in r_1 by itself and then substituting the last four recursive predicates p by the exit predicate q and the p predicate with s in the body, while rule r''_{53} is obtained by expanding both the second and last recursive predicates p in the body of rule r_1 by r_1 itself and then substituting all recursive predicates except the first one (which is substituted by s) by the exit predicate q. r''_{6i} and r''_{7i} are obtained in the same way.

5 General ZYT-linearizability of an n-linear sirup

So far we have considered the problem of whether a given n-linear program is equivalent to its derivative. The derivative of an n-linear program essentially consists of n linear rules. However, when the number of linear rules in a program is large, the computation could be very inefficient. Therefore, it is still interesting to determine whether an n-linear program is equivalent to a linear sirup when n is large. For example, given the following n-linear program,

$\wp$:
$r_0 : p(X_1, X_2, \ldots, X_k) \text{ :- } q(X_1, X_2, \ldots, X_k).$
$r_1 : p(X_1, X_2, \ldots, X_k) \text{ :- } p(Y_1, Y_2, \ldots, Y_k), \cdots,$
$p(Z_1, Z_2, \ldots, Z_k), p(U_1, U_2, \ldots, U_k).$

we want to find whether it is equivalent to the following linear sirup:

$\wp'$:
$r'_0 : p(X_1, X_2, \ldots, X_k) \text{ :- } q(X_1, X_2, \ldots, X_k).$
$r'_1 : p(X_1, X_2, \ldots, X_k) \text{ :- } q(Y_1, Y_2, \ldots, Y_k), \cdots,$
$q(Z_1, Z_2, \ldots, Z_k), p(U_1, U_2, \ldots, U_k).$

where the linear recursive rule is obtained from r_1 by substituting predicate q for all predicates p except the last one (in fact, any predicate p can be chosen). Recall that $\wp'$ is called the general ZYT-linearization of $\wp$. Moreover, when $\wp \equiv \wp'$, we say that $\wp$ is *general ZYT-linearizable*. R. Ramakrishnan *et al.* [1993] considered the case of $n = 2$. We now generalize their result to the case of n-linear programs with $n \geq 3$. In Section 5.1 we shall show a tighter condition which is necessary and sufficient to determine this equivalence.

Theorem 5.1: It is *NP*-hard to determine whether a sirup is general ZYT-linearizable.

Proof: We first construct an n-linear program $\wp$ as follows:

$\wp$:
$r_0 : p(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$X_{n1}, X_{n2}, X_{n3}, X_{n4}) \text{ :- }$
$p_0(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$X_{n1}, X_{n2}, X_{n3}, X_{n4}).$
$r_1 : p(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$X_{n1}, X_{n2}, X_{n3}, X_{n4}) \text{ :- }$
$p(X_{11}, X_{12}, X_{13}, Y_{14}, \ldots, -, -, -, -),$
$\varphi_G(Y_{14}, X_{14}),$
$p(X_{21}, X_{22}, X_{23}, Y_{24}, \ldots, -, -, -, -),$
$\varphi_G(Y_{24}, X_{24}),$
$\ldots$
$p(X_{n1}, X_{n2}, X_{n3}, Y_{n4}, \ldots, -, -, -, -),$
$\varphi_G(Y_{n4}, X_{n4}).$

where φ_G is the EDB predicate in r_G, the rule defined in Section 2.1. We also have the following program $\wp'$, the general ZYT-linearization of $\wp$:

$\wp'$:
$r'_0 : p(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$X_{n1}, X_{n2}, X_{n3}, X_{n4}) \text{ :- }$
$p_0(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$X_{n1}, X_{n2}, X_{n3}, X_{n4}).$
$r'_1 : p(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$X_{n1}, X_{n2}, X_{n3}, X_{n4}) \text{ :- }$
$p_0(X_{11}, X_{12}, X_{13}, Y_{14}, \ldots, -, -, -, -),$
$\varphi_G(Y_{14}, X_{14}),$
$p_0(X_{21}, X_{22}, X_{23}, Y_{24}, \ldots, -, -, -, -),$
$\varphi_G(Y_{24}, X_{24}),$
$\ldots$
$p_0(X_{(n-1)1}, X_{(n-1)2}, X_{(n-1)3}, Y_{(n-1)4},$
$\ldots, -, -, -, -), \varphi_G(Y_{(n-1)4}, X_{(n-1)4}),$
$p(X_{n1}, X_{n2}, X_{n3}, Y_{n4}, \ldots, -, -, -, -),$
$\varphi_G(Y_{n4}, X_{n4}).$

Then we can prove that $\wp$ is general ZYT-linearizable iff r_G is 1-bounded using the same technique as in the proof of Theorem 3.1. □

5.1 Additional Conditions for general ZYT-linearizability of trilinear sirups

As in the previous sections, given an n-linear sirup $\wp$ there are linear programs $\wp_2$ and $\wp_3$ such that $\wp$ is general ZYT-linearizable if and only if $\wp_1 \equiv_p \wp_2$ or $\wp_1 \equiv \wp_3$, where $\wp_1$ is the general ZYT-linearization of $\wp$. We also have a sufficient condition for testing the general ZYT-linearizability using $\wp_3$, as the one given in Theorem 3.3 for differentiability. For the case $n = 3$, if $\wp$ is the following program, $\wp_2$ and $\wp_3$ are shown next.

$\wp$:
$r_0 : p(\vec{X}) \text{ :- } q(\vec{X}).$
$r_1 : p(\vec{X}) \text{ :- } p(\vec{Y}), p(\vec{Z}), p(\vec{U}), G.$

$\wp'$:
$r'_0 : p(\vec{X}) \text{ :- } q(\vec{X}).$
$r'_1 : p(\vec{X}) \text{ :- } q(\vec{Y}), q(\vec{Z}), p(\vec{U}), G.$

$\wp''$:
$r''_0 : s(\vec{X}) \text{ :- } q(\vec{X}).$
$r''_1 : s(\vec{X}) \text{ :- } q(\vec{Y}), q(\vec{Z}), s(\vec{U}), G.$
$r''_2 : p(\vec{X}) \text{ :- } s(\vec{X}).$

$r''_3 : p(\vec{X}) \text{ :- } q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G',$
$\quad q(\vec{Z}), s(\vec{U}), G.$
$r''_4 : p(\vec{X}) \text{ :- } q(\vec{Y}), q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G', s(\vec{U}), G.$
$r''_5 : p(\vec{X}) \text{ :- } q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G',$
$\quad q(\vec{Y''}), q(\vec{Z''}), q(\vec{U''}), G'', s(\vec{U}), G.$

$\wp''' :$
$r'''_0 : p(\vec{X}) \text{ :- } q(\vec{X}).$
$r'''_1 : p(\vec{X}) \text{ :- } q(\vec{Y}), q(\vec{Z}), p(\vec{U}), G.$
$r'''_2 : p(\vec{X}) \text{ :- } q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G',$
$\quad q(\vec{Z}), p(\vec{U}), G.$
$r'''_3 : p(\vec{X}) \text{ :- } q(\vec{Y}), q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G',$
$\quad p(\vec{U}), G.$
$r'''_4 : p(\vec{X}) \text{ :- } q(\vec{Y'}), q(\vec{Z'}), q(\vec{U'}), G',$
$\quad q(\vec{Y''}), q(\vec{Z''}), q(\vec{U''}), G'', p(\vec{U}), G.$

where: $\wp'''$ is a program obtained from $\wp''$ by substituting predicate p for predicate s; r''_3 is obtained by expanding the first recursive predicate p of rule r_1 by r_1 itself and substituting the first four recursive predicates by the exit predicate q in the basis rule r_0, and finally the predicate p in the expanded rule is replaced by s; rule r''_4 is obtained similarly by expanding the second recursive predicate p; rule r''_5 is obtained by expanding the first recursive predicate and the second recursive predicate p simultaneously in the rule r_1 by r_1 itself, and finally all predicates p except the last one, which is replaced by s, are replaced by q.

Given a program $\wp$ as follows,
hangindent=0.1in$\wp$:
$r_0 : p(X, Y) \text{ :- } e(X, Y).$
$r_1 : p(X, Y) \text{ :- } p(X, Z_1), p(Z_1, Z_2), \ldots, p(Z_{n-1}, Y).$

we can use the results in this section to show that it is equivalent to its general ZYT-linearization:

$\wp' :$
$r'_0 : p(X, Y) \text{ :- } e(X, Y).$
$r'_1 : p(X, Y) \text{ :- } e(X, Z_1), e(Z_1, Z_2), \ldots,$
$\quad e(Z_{n-2}, Z_{n-1}), p(Z_{n-1}, Y).$

It is well known that the bilinear program to find the transitive closure is ZYT-linearizable [Ullman, 1989]. From the above discussion, therefore, all transitive programs to find all paths composed of either unit paths or paths composed of n subpaths can be expressed by a linear sirup.

6 k-ZYT-linearizability of an n-linear sirup

In this section, we consider the k-ZYT linearization of an n-linear sirup $\wp$ which is obtained from $\wp$ by substituting the exit predicate in the nonrecursive rule for the first k IDB predicates in the recursive rule ($1 \leq k < n$). This linearization is more general than both the ZYT-linearization and the general ZYT-linearization. In the following theorem, we prove that the problem of determining whether an n-linear sirup is k-ZYT-linearizable is NP-hard. Furthermore, we shall show a tighter sufficient and necessary condition to test k-ZYT-linearizability of an n-linear sirup.

Theorem 6.1: It is NP-hard to determine whether an n-linear sirup is k-ZYT-linearizable.

Proof: Let r_G be the rule defined in Section 2.1. We first construct the following n-linear program, where φ_G is the EDB predicate in r_G's body:

$\wp :$
$r_0 : p(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24},$
$\quad \ldots, X_{n1}, X_{n2}, X_{n3}, X_{n4}) \text{ :- }$
$\quad p_0(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$\quad X_{n1}, X_{n2}, X_{n3}, X_{n4}).$
$r_1 : p(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$\quad X_{n1}, X_{n2}, X_{n3}, X_{n4}) \text{ :- }$
$\quad p(X_{11}, X_{12}, X_{13}, Y_{14}, \ldots, -, -, -, -),$
$\quad \varphi_G(Y_{14}, X_{14}),$
$\quad \ldots$
$\quad p(X_{k1}, X_{k2}, X_{k3}, Y_{k4}, \ldots, -, -, -, -),$
$\quad \varphi_G(Y_{k4}, X_{k4}),$
$\quad p(X_{(k+1)1}, X_{(k+1)2}, X_{(k+1)3}, Y_{(k+1)4},$
$\quad \ldots, -, -, -, -), \varphi_G(Y_{(k+1)4}, X_{(k+1)4}),$
$\quad \ldots$
$\quad p(X_{n1}, X_{n2}, X_{n3}, Y_{n4}, \ldots, -, -, -, -),$
$\quad \varphi_G(Y_{n4}, X_{n4}).$

The k-ZYT-linearization of $\wp$ is:

$\wp' :$
$r'_0 : p(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$\quad X_{n1}, X_{n2}, X_{n3}, X_{n4}) \text{ :- }$
$\quad p_0(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$\quad X_{n1}, X_{n2}, X_{n3}, X_{n4}).$
$r'_1 : p(X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, \ldots,$
$\quad X_{n1}, X_{n2}, X_{n3}, X_{n4}) \text{ :- }$
$\quad p_0(X_{11}, X_{12}, X_{13}, Y_{14}, \ldots, -, -, -, -),$
$\quad \varphi_G(Y_{14}, X_{14}),$
$\quad \ldots$
$\quad p_0(X_{k1}, X_{k2}, X_{k3}, Y_{k4}, \ldots, -, -, -, -),$
$\quad \varphi_G(Y_{k4}, X_{k4}),$
$\quad p(X_{(k+1)1}, X_{(k+1)2}, X_{(k+1)3}, Y_{(k+1)4},$
$\quad \ldots, -, -, -, -), \varphi_G(Y_{(k+1)4}, X_{(k+1)4}),$
$\quad \ldots$
$\quad p(X_{n1}, X_{n2}, X_{n3}, Y_{n4}, \ldots, -, -, -, -),$
$\quad \varphi_G(Y_{n4}, X_{n4}).$

Again, we can prove that $\wp$ is k-ZYT-linearizable iff r_G is 1-bounded using the same technique as in the proof of Theorem 3.1. □

6.1 Additional Conditions for k-ZYT-linearizability of n-linear sirups

Given an n-linear sirup $\wp$, there are programs $\wp_2$ and $\wp_3$ which are more "linear" than $\wp$; that is, rules in $\wp_2$ and $\wp_3$ contain smaller number of recursive predicates than rules in $\wp$, such that $\wp$ is k-ZYT-linearizable, i.e., equivalent to its k-ZYT-linearization $\wp_1$, if and only if $\wp_1 \equiv_p \wp_2$ or $\wp_1 \equiv \wp_3$. Also we have the resut that $\wp$ is k-ZYT-linearizable iff it is uniformly equialent to $\wp_3$. Those progrmas for the case $k = 2$ are shown below.

$\wp :$
$r_0 : p(\vec{X}) \text{ :- } q(\vec{X}).$
$r_1 : p(\vec{X}) \text{ :- } p(\vec{X_1}), p(\vec{X_2}), \ldots, p(\vec{X_n}), G.$

$\wp_1 :$
$r'_0 : p(\vec{X}) \text{ :- } q(\vec{X}).$

$r'_1 : p(\vec{X}) \text{ :- } q(\vec{X_1}), q(\vec{X_2}), p(\vec{X_3}),$
$\quad \ldots, p(\vec{X_n}), G.$

$\wp_2$:

$r''_0 : s(\vec{X}) \text{ :- } q(\vec{X}).$
$r''_1 : s(\vec{X}) \text{ :- } q(\vec{X_1}), q(\vec{X_2}),$
$\quad s(\vec{X_3}), \ldots, s(\vec{X_n}), G.$
$r''_2 : p(\vec{X}) \text{ :- } s(\vec{X}).$
$r''_3 : p(\vec{X}) \text{ :- } q(\vec{X'_1}), q(\vec{X'_2}), \ldots, q(\vec{X'_n}), G',$
$\quad q(\vec{X_2}), s(\vec{X_3}), \ldots, s(\vec{X_n}), G.$
$r''_4 : p(\vec{X}) \text{ :- } q(\vec{X_1}), q(\vec{X'_1}), q(\vec{X'_2}), \ldots,$
$\quad q(\vec{X'_n}), G', s(\vec{X_3}), \ldots, s(\vec{X_n}), G.$
$r''_5 : p(\vec{X}) \text{ :- } q(\vec{X'_1}), q(\vec{X'_2}), \ldots, q(\vec{X'_n}), G',$
$\quad q(\vec{X''_1}), q(\vec{X''_2}), \ldots, q(\vec{X''_n}), G'',$
$\quad s(\vec{X_3}), \ldots, s(\vec{X_n}), G.$

$\wp_3$:

$r'''_0 : p(\vec{X}) \text{ :- } q(\vec{X}).$
$r'''_1 : p(\vec{X}) \text{ :- } q(\vec{X_1}), q(\vec{X_2}), p(\vec{X_3}),$
$\quad \ldots, p(\vec{X_n}), G.$
$r'''_2 : p(\vec{X}) \text{ :- } q(\vec{X'_1}), q(\vec{X'_2}), \ldots, q(\vec{X'_n}), G',$
$\quad q(\vec{X_2}), p(\vec{X_3}), \ldots, p(\vec{X_n}), G.$
$r'''_3 : p(\vec{X}) \text{ :- } q(\vec{X_1}), q(\vec{X'_1}), q(\vec{X'_2}), \ldots, q(\vec{X'_n}), G',$
$\quad p(\vec{X_3}), \ldots, p(\vec{X_n}), G.$
$r'''_4 : p(\vec{X}) \text{ :- } q(\vec{X'_1}), q(\vec{X'_2}), \ldots, q(\vec{X_n}), G',$
$\quad q(\vec{X''_1}), q(\vec{X''_2}), \ldots, q(\vec{X''_n}), G'',$
$\quad p(\vec{X_3}), \ldots, p(\vec{X_n}), G.$

Where: r''_3 is obtained by expanding the first recursive predicate p of rule r_1 in $\wp$ and replacing the first $n+1$ recursive predicates p by the exit predicate q in the basis rule r_0, and finally the $n-2$ left-most occurrences of predicate p in the body of the expanded rule is replaced by s; r''_4 is obtained similarly by expanding the second recursive predicate p by rule r_1; r''_5 is obtained by expanding the first recursive predicate p and the second recursive predicate p simultaneously in the rule r_1 by r_1 itself in $\wp$, and finally all the first $2n$ recursive predicates p are replaced by q and the last $n-2$ recursive predicates p are replaced by s.

7 Conclusions

In this paper, we have studied three more general linearizations of an n-linear sirup for arbitrary n: derivative, general ZYT-linearization, k-ZYT-linearization. The derivative and general ZYT-linearization generalize the ZYT-linearization for the more general case considered by R. Ramakrishnan *et al.*, while the k-ZYT-linearization is a generalization of the general ZYT-linearization.

We have shown that the problem of deciding the possibility of all above linearizations for an n-linear sirup is *NP*-hard; we do not know if they are decidable. In order to test whether an n-linear sirup is differentiable, general ZYT-linearizable, or k-ZYT-linearizable, we showed a tighter condition respectively which is both necessary and sufficient instead of directly testing whether the given n-linear sirup is equivalent to one of the above specific linearizations. Moreover, this tighter condition gives us the possibility to test linearizability by just testing the uniform equivalence between two linear programs.

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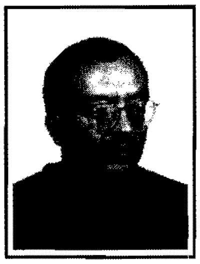

__Héctor J. Hernández__ received his Bachelor, Master, and Ph.D. degrees all in Computer Science from Monterrey's Institute of Technology (ITESM Campus Monterrey), University of Waterloo, and University of Alberta, respectively. He was an Assistant Professor in the departments of Computer Science of Texas A&M University and New Mexico State University. He is a Professor at Univsidad Autónoma de Cd. Juárez. His research interests are in database management, particulary in query processing.

__Dongxing Tang__ received his Ph.D. in Computer Science from New Mexico State University. He worked as a professional computer consultant for VLSI design analysis tools in IBM and network management in AT&T from 1995 to 1997. He is currently working in UTStarcom, Inc. as a computer specialist in the area of telecommunication. He is a member of the Laboratory for Logic, Databases, and Advanced Programming. His research interests include database system, logic programming and parallel processing.