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Setting Cournot Versus Lyapunov Games Stability Conditions and Equilibrium Point Properties

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In potential games, the best-reply dynamics results in the existence of a cost function such that each player's best-reply set equals the set of minimizers of the potential given by the opponents' strategies. The study of sequential best-reply dynamics dates back to Cournot and, an equilibrium point which is stable under the game's best-reply dynamics is commonly said to be Cournot stable. However, it is exactly the best-reply behavior that we obtain using the Lyapunov notion of stability in game theory. In addition, Lyapunov theory presents several advantages. In this paper, we show that the stability conditions and the equilibrium point properties of Cournot and Lyapunov meet in potential games.

Keywords: Cournot; Lyapunov; potential games; dominance-solvable games; routing games; shortest-path; best-reply.

Subject Classification: 22E46, 53C35, 57S20

1. Introduction

1.1. *Brief review*

Potential games were introduced in the seminal work of Monderer and Shapley [1996]. Various notions of potential games have been introduced and studied in the literature. Voorneveld [2000] introduced the best-reply potential games having the distinctive feature that it allows infinite improvement paths, by imposing restrictions only on paths in which players can improve actually deviate to a best-reply. The notions of pseudo-potential games were studied by Dubey *et al.* [2006]. All these classes of potential games start with an arbitrary strategy profile, and using a

Table 1. Potential game with Cournot cycles.

	A	B	C
A	$(3, 3)$	$\underline{(\mathbf{4}, \mathbf{5})}$	$\underline{(\mathbf{5}, \mathbf{10})}$
B	$\underline{(\mathbf{5}, \mathbf{4})}$	$(5, 5)$	$\underline{(\mathbf{10}, \mathbf{3})}$
C	$\underline{(\mathbf{10}, \mathbf{5})}$	$\underline{(\mathbf{3}, \mathbf{10})}$	$(3, 3)$

single real-valued function on the strategy space a player who can improve deviates to a better strategy. The iteration process converges to a Nash equilibrium point. It is important to note that potential games definition is not free of cycles. It is possible to construct a utility function that satisfies the properties established by Monderer and Shapley [1996] of potential games with strictly increasing differences that admits Cournot cycles (see Table 1).

This example motivates the introduction of the Lyapunov games where cycles are avoided. We will prove that a Lyapunov game is free of Cournot cycles (see Theorem 1). Moreover, the introduction of a Lyapunov-like function as a cost function satisfies all the definitions of potential games [Monderer and Shapley, 1996; Voorneveld, 2000; Dubey *et al.*, 2006] and presents several advantages [Kalman and Bertram, 1960]: (a) a natural existence of the equilibrium point should be ensured by definition, (b) a Lyapunov-like function can be constructed to respect the constraints imposed by the game, (c) a Lyapunov-like function definitely converges to a Lyapunov equilibrium point, and (d) a Lyapunov equilibrium point presents properties of stability that are not necessarily present in a different kind of equilibrium point (i.e., Nash and Cournot). By definition, a Lyapunov-like function monotonically decreases and converges to a Lyapunov equilibrium point. The best-reply dynamics results in a natural implementation of the behavior of a Lyapunov-like function. As a result, a Lyapunov game has also the benefit that it is a common knowledge of the players that only best-reply are chosen. In Lyapunov games, the best-reply evolution of an arbitrary strategy profile, given by a Lyapunov-like function, reaches a minimum or approaches to an infimum. Clempner [Clempner and Poznyak, 2011, 2013] showed that the evolution of a Lyapunov-like function converges to a Nash equilibrium.

Lyapunov games embrace many practical application domains including potential games, dominance-solvable games, routing games and shortest-path games [Engelberg and Schapira, 2011; Fabrikant and Papadimitriou, 2008; Fabrikant *et al.*, 2013]. In general, all the classes of Potential games reported in the literature are contained into the definition of Lyapunov games.

1.2. *Main results*

In this paper, we propose a powerful but simple approach to study that a Lyapunov equilibrium point coincides with the Cournot equilibrium point and the stability conditions in noncooperative games. The main idea of this work is to exploit the order and monotonicity properties of the Lyapunov-like functions of the game using a lattice theoretical approach. The lattice structure provides an order structure on the equilibrium set and some stability properties [see Clempner and Poznyak, 2011]. Then, we are able, in the first place, to obtain results regarding that a Lyapunov game is free of Cournot cycles and to establish that a Lyapunov game is a best-reply potential game. In this sense, an equilibrium point which is stable under the game's best-reply dynamics is Cournot stable. Following, we provide evidence that a Lyapunov-like function has an asymptotically approached infimum or reaches a minimum and that the Lyapunov equilibrium point is the sink of the game. For the main result of the paper, we prove that a Lyapunov equilibrium point coincides with the Cournot–Nash equilibrium point. We demonstrate that the best-reply dynamics of Cournot is represented naturally by the behavior of a Lyapunov-like function and that the Cournot stabilility it is exactly the best-reply behavior that we obtain using the Lyapunov notion of stability in game theory. Finally, we show that Lyapunov games are weakly acyclic and that all the stability conditions established by the weak acyclicity games are naturally satisfied by the Lyapunov games.

1.3. *Organization of the paper*

A game description, along with some notational matters, is given in Sec. 2. Subsequently, the Lyapunov game definition, acyclic properties and its relationship with potential games are presented in Sec. 3. Section 4 establishes the Lyapunov–Cournot stability conditions and equilibrium point properties. In Sec. 5, we show the connection between Lyapunov games and weak acyclic games. Finally, the results of the paper are summarized and directions of further research are discussed in Sec. 6.

2. Mathematical Preliminaries

The aim of this section is to present the mathematical background needed to understand the rest of the paper.

A *noncooperative game* is a tuple $G = \langle \mathcal{N}, (S_\iota)_{\iota\in\mathcal{N}}, (\leq_\iota)_{\iota\in\mathcal{N}} \rangle$ where $\mathcal{N} = \{1, 2, \ldots, n\}$ is a finite set of players; S_ι a finite set of "pure" strategies (henceforth called actions) of each player $\iota \in \mathcal{N}$; $(\leq_\iota)$ is a binary relationship over $S := \prod_{\iota\in\mathcal{N}} S_\iota$ reflecting the preferences of the player ι over the outcomes.

It is assumed that the relation $(\leq_\iota)$ establishes a poset on S, i.e., given $r, s, t \in S$ we expect the preference relation $(\leq_\iota)$ to be fulfilled, and the following axioms hold: reflexivity ($r \leq_\iota r$), antisymmetry ($r \leq_\iota s$ and $s \leq_\iota r$ implies that $r = s$), transitivity ($r \leq_\iota s$ and $s \leq_\iota t$ implies that $r \leq_\iota t$).

Although the preference relation is the basic primitive of any decision problem (and generally observable), it is much easier to work with a consistent *cost function*

$$\mathbb{U}_\iota : S \to \mathbb{R}^n_+ \tag{1}$$

because we only have to use n real numbers.