Computing The Strong Nash Equilibrium For Markov Chains Games

Julio B. Clempner[1,]

Center for Economics, Management and Social Research, National Polytechnic Institute, Lauro Aguirre 120, col. Agricultura, Del. Miguel Hidalgo, 11360, Mexico City, Mexico

Alexander S. Poznyak

Department of Control Automatics, Center for Research and Advanced Studies, Av. IPN 2508, Col. San Pedro Zacatenco, 07360 Mexico City, Mexico

Abstract

In this paper we present a novel method for finding the strong Nash equilibrium. The approach consists on determining a scalar λ^* and the corresponding strategies $d^*(\lambda^*)$ fixing specific bounds (*min* and *max*) that belong to the Pareto front. Bounds correspond to restrictions imposed by the player over the Pareto front that establish a specific decision area where the strategies can be selected. We first exemplify the Pareto front of the game in terms of a nonlinear programming problem adding a set of linear constraints for the Markov chain game based on the c-variable method. For solving the strong Nash equilibrium problem we propose to employ the Euler method and a penalty function with regularization. The Tikhonov's regularization method is used to guarantee the convergence to a single (strong) equilibrium point. Then, we established a nonlinear programming method to solve the successive single-objective constrained problems that arise from taking the regularized functional of the game. To achieve the goal, we implement the gradient method to solve the first-order optimality conditions. Starting from an *utopia point* (Pareto optimal point) given an initial λ of the individual objectives the method solves an optimization problem adding linear constraints required to find the optimal strong strategy $d^*(\lambda^*)$. We show that in the regularized problem the functional of the game decrease and finally converges, proving the existence and uniqueness of strong Nash equilibrium (Pareto-optimal Nash equilibrium). In addition, we present the convergence conditions and compute the estimate rate of convergence of variables γ and δ corresponding to the step size parameter of the gradient method and the Tikhonov's regularization respectively. Moreover, we provide all the details needed to implement the method in an efficient and numerically stable way. The usefulness of the method is successfully demonstrated by a numerical example.

Keywords: strong Nash equilibrium, Pareto-optimal Nash equilibrium, Markov chains, game theory.

1. Introduction

1.1. Brief review

In strategic games the analysis of Nash equilibrium has captured a central place in game theory since its beginning. A Nash game is a solution concept of a non-cooperative game in which no player can unilaterally change her/his strategy to increase her/his payoff [1–4]. However, in many situations players have an incentive to cooperate, because cooperation motivates to dramatically improve the circumstances of every participant. In these cases, the non-cooperation in Nash equilibrium significantly damages the conceptualization of the game in practice: Nash equilibrium is only resilient against unilateral deviations. In addition, Nash equilibrium presents several important problems. The main difficulty is that usually many games are ill with multiple equilibria.

Email addresses: julio@clempner.name (Julio B. Clempner), apoznyak@ctrl.cinvestav.mx (Alexander S. Poznyak)

[1] corresponding author

Solution to this problem is a refinement of Nash equilibria. Many proposals have been suggested in the literature without convincing results [5–8]. However, Pareto-optimal Nash or Strong Nash equilibrium (SNE) [9] is one of the most interesting equilibrium concepts which ensure a more restrictive concept than the Nash equilibrium. While the Nash concept of stability defines equilibrium only in terms of unilateral deviations, in a strong equilibrium there is no strategic profile for which a subset of players has a joint deviation that strictly benefits all of them. In view of the fact that no coalition of players that can improve their payoffs by collective unilateral deviation, a SNE is a Nash equilibrium that it is also Pareto efficient among the Nash equilibria [9]. A Pareto efficient (or optimal) strategy is a situation in which no player can improve his/her payoff without decreasing the payoff of someone else. Thus, SNE presents the advantages of a cooperative behavior in a non-cooperative environment.

Several important points come out when dealing with SNE: 1) SNE not necessarily exist for all games; however, in this paper we introduced the Tikhonov's regularization method that permit the existence of a unique strong Nash equilibrium; 2) the rate of convergence of the regularization parameter chosen to control the size of the solution vector; 3) the computational complexity of SNE to determine whether the concept is reasonable from a computational point of view, 4) the computational complexity related to computing the power set of the players containing all possible player coalitions along the Pareto front.

However, proving the existence of SNE is a difficult problem [10] and there are a small number of computational tools available for finding the SNE. Conitzer and Sandholm [11], Gatti et al. [12] and, Hoefer and Skopalik [13] prove that the complexity of computing a SNE is known to be NP-complete problem. In the literature, can be found several algorithms to search strong Nash equilibria for specific classes of games, e.g., congestion games [13–18], connection games [19, 20], maxcut games [21], voting models [22–24], coalition formation [25–29], other fields [30–40]. Properties, existence conditions, and an analytical algorithm are described in [10]. However, how to effectively find the SNE is still unknown to computer scientists. This fact makes it meaningful to develop new algorithms to compute Strong Nash Equilibrium.

1.2. Motivation

In game theory problem there are given n players function $J^1, ..., J^n : \mathbb{R}^n \to \mathbb{R}$ which have to be minimized:

$$\mathbf{J}(d) = (J^1(d), ..., J^n(d)) \to \min_d$$

over the class of all admissible policies (strategies) d (see Section 2 for details). If d^* minimizes $\mathbf{J}(d)$ in the sense of Pareto, then d^* is said to be a *Pareto policy*. The fundamental optimality concept is that of a local Pareto optimal policy, which is a policy such that no improvement in all the objectives can be achieved by moving to a neighboring feasible point. The game theory problem is different from single-objective nonlinear programming because the set of Pareto optimal points are usually a continuum that may have disjoint components.

Pareto optimality defines a partial ordering of strategy profiles. The ordering can be captured by formulating through a multi-objective problem, since it is nature to connect the partial ordering of Pareto optimal to the concept of optimal in multi-objective scenario. The problem that comes up is how to compute all the optimal compromises of this multi-objective optimization problem.

The existence of Pareto policies can be obtained via the scalarization approach, in which the Markov chain game is reduced to a single-objective with a weighted objective function of the form

$$\lambda^\top \mathbf{J} = (\lambda, \mathbf{J}) = \lambda^1 J^1(d) + ... + \lambda^n J^n(d) \quad (1)$$

for some vectors λ in the nonnegative orthant R^n_+.

We exemplify the Pareto front of the game in terms of a nonlinear programming problem. In addition, we introduce a set of linear constraints for the Markov chain game based on the c-variable method. An advantage of this approach is that the formulation of the problem is easy to conceptualize and makes the problem computationally tractable. With this class of constraints we are able to formulate a nonlinear programming formulation for the having a fast way to determine the Pareto front. We present a numerical experiment for which the necessary condition of a minimum of the game is solved using a Gradient Projection method. As a result we have a full discrete approximation of the Pareto front based on a new procedure which leads to a simple and logically justified computational realization.